From Smith to Jordan

A Module-Theoretic Path Through Linear Algebra

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Similar Matrices

A problem from linear algebra

Let $M := \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A := \begin{pmatrix} 1482 & -957 & -210 & -49 \\ 3396 & -2193 & -481 & -112 \\ -6102 & 3940 & 863 & 200 \\ 4657 & -3005 & -657 & -152 \end{pmatrix}$.

Question: Are M and A similar? $\exists P \in \mathrm{GL}(4, \mathbb{Q}) : A = P^{-1}MP$?

Answer: `JordanDecomposition[A][[2]]` and `JordanDecomposition[M][[2]]` both yield the result:

$$J = \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix},$$

hence there is $P \in \mathrm{GL}(4, \mathbb{C})$ with $A = P^{-1}MP$.

Question: Is there such a $P \in \mathrm{GL}(4, \mathbb{Q})$?

Smith Normal Form

Smith Normal Form

Theorem (H. Smith 1861). Let $A \in \mathbb{Z}^{n\times n}$. Then there exist $U, S, V \in \mathbb{Z}^{n\times n}$ such that

- $S = UAV$,
- $\det(U)$ and $\det(V)$ are in $\{-1, 1\}$ $\quad$ (U, V are unimodular)
- $S = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & d_n \end{pmatrix}$ with $d_1 \mid d_2 \mid \cdots \mid d_n$ and all $d_i \geq 0$.

Example

$$\begin{pmatrix} 6 & 0 \\ 0 & 24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 8 & -1 \end{pmatrix} \cdot \begin{pmatrix} 18 & 30 \\ 120 & 192 \end{pmatrix} \cdot \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

Computation of the Smith Normal Form

Use the following operations

1. swap rows
2. add multiple of one row to another row
3. swap columns
4. add multiple of one column to another column
5. multiply a row with -1

to get to the desired form. Instead of applying this procedure to $\begin{pmatrix} 18 & 30 \\ 120 & 192 \end{pmatrix}$, apply it to the first 2 rows and colums of

$$\begin{pmatrix} 18 & 30 & 1 & 0 \\ 120 & 192 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Computing the Smith Normal Form

All of these transformations are multiplications from left or right with unimodular matrices.
U, V can be revovered because:

$$\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} UAV & U \\ V & 0 \end{pmatrix}.$$

Computing the Smith Normal Form

We compute the SNF of $\begin{pmatrix} 18 & 30 \\ 120 & 192 \end{pmatrix}$.

$$\begin{pmatrix} 18 & 30 & 1 & 0 \\ 120 & 192 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 18 & 12 & 1 & 0 \\ 120 & 72 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 18 & 12 & 1 & 0 \\ 12 & 0 & -6 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Hence

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The loop invariant of computing the SNF

The transformation $A \rightsquigarrow (U, S, V)$ makes sense if A and S have some properties in common.

Theorem. Let S be the Smith normal form of $A \in \mathbb{Z}^{n \times n}$. Then:

1. A and S are unimodularly equivalent: there are invertible U, V with $UAV = S$.

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   Proof: $[x]_{\operatorname{col}(A)} \mapsto [Ux]_{\operatorname{col}(S)}$ is an isomorphism.

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   This characterizes finitely generated abelian groups:

   - If $G = \langle g_1, \ldots, g_n \rangle$, then $\varphi : \mathbb{Z}^n \to G$, $\varphi(z_1, \ldots, z_n) := z_1 * g_1 + \cdots + z_n * g_n$ is an epimorphism.
   - $\ker(\varphi)$ is a submodule of $\mathbb{Z}^n$, hence equal to $\operatorname{col}(A)$ for some $A \in \mathbb{Z}^{n \times n}$.
   - Thus $G \cong \mathbb{Z}^n / \operatorname{col}(S)$ with $S = \operatorname{diag}(d_1, \ldots, d_n)$.
   - Hence $G \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n}$ (H. Poincaré 1900).

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3. For each $k \in \mathbb{N}$, the gcd of the determinants of all $k \times k$-submatrices is the same for A and S.

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   Follows from the Cauchy-Binet formula for determinants: For $A \in \mathbb{Z}^{n \times m}, B \in \mathbb{Z}^{m \times n}$,

$$\det(AB) = \sum_{I \subseteq \underline{m}, |I| = n} \det(A|_{\underline{n} \times I}) \det(B|_{I \times \underline{n}}).$$

Smith Normal Form is normal

Theorem. Let $A, B \in \mathbb{Z}^{m \times n}$, and let S_A and S_B be their Smith normal forms. TFAE:

1. A is unimodularly equivalent to B.
2. The modules $\mathbb{Z}^m / \operatorname{col}(A)$ and $\mathbb{Z}^m / \operatorname{col}(B)$ are isomorphic.
3. $S_A = S_B$.

Key observation: In the module $M = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$ with $d_1 \mid d_2 \mid \cdots \mid d_k$, we have

$$(d_{k-s}) = \{x \in \mathbb{Z} \mid xM \text{ can be generated by } \leq s \text{ elements}\}.$$

Similarity of Matrices

The module V_A

With $A \in \mathbb{K}^{n\times n}$, we associate the $\mathbb{K}[t]$-module $V_A := \mathbb{K}^n$. The module operation is defined by

$$t \star_A v := A \cdot v, \quad (\alpha t^0) \star_A v := \alpha * v \;\text{ for } \alpha \in \mathbb{K}, v \in \mathbb{K}^n.$$

and hence

$$p \star_A v = (\sum_{i=0}^{n} p_i\, t^i) \star_A v = \sum_{i=0}^{n} p_i * (A^i \cdot v) = \hat{p}(A) \cdot v.$$

Theorem. A and B are similar over $\mathbb{K}$ $\iff$ V_A and V_B are isomorphic $\mathbb{K}[t]$-modules.

The module V_A

The $\mathbb{K}[t]$-module V_A is generated by $\boldsymbol{e}_1, \dots \boldsymbol{e}_n$. Hence

$$\varphi : \mathbb{K}[t]^n \to V_A,\ \Phi(\boldsymbol{e}_i) := \boldsymbol{e}_i$$

can be extended to the module epimorphism

$$\Phi(p_1, \dots, p_n) := \sum_{i=1}^{n} \hat{p_i}(A) \cdot \boldsymbol{e}_i.$$

The Fundamental Theorem.

$$\ker(\Phi) = \mathrm{col}(t * I - A).$$

The module V_A

Recall: $\Phi(p_1, \ldots, p_n) := \sum_{i=1}^{n} \hat{p}_i(A) \cdot \boldsymbol{e}_i$.

We prove $\mathrm{col}(t * I - A) \subseteq \ker(\Phi)$: Let

$$s_j = t * \boldsymbol{e}_j - (a_{1,j}, a_{2,j}, \ldots, a_{n,j})$$

be the j th column of $t * I - A$. Then

$$\Phi(s_j) = A \cdot \boldsymbol{e}_j - \sum_{i=1}^{n} a_{i,j} \boldsymbol{e}_i,$$

and the i-th entry of this vector is $a_{i,j} - a_{i,j} = 0$.

$\mathrm{col}(t * I - A) \supseteq \ker(\Phi)$: Follows from $\dim_{\mathbb{K}}(\mathbb{K}[t]^n / \mathrm{col}(t * I - A)) \leq n$ because every $(p_1, \ldots, p_n)$ can be reduced modulo $\mathrm{col}(t * I - A)$ to $(\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$.

The fundamental isomorphism

Theorem. For $A \in \mathbb{K}^{n \times n}$, $V_A \cong K[t]^n / \operatorname{col}(t * I - A)$.

Hence: A and B are similar over $\mathbb{K}$ $\Leftrightarrow$ V_A and V_B are isomorphic $\mathbb{K}[t]$-modules $\Leftrightarrow$ $K[t]^n / \operatorname{col}(t * I - A) \cong K[t]^n / \operatorname{col}(t * I - B)$.

Obeservations:

- For matrices over $\mathbb{Z}$, we can solve the problem whether $\mathbb{Z}^n / \operatorname{col}(M) \cong \mathbb{Z}^n / \operatorname{col}(N)$ via the Smith Normal Form.
- $K[t]$ is "similar" to $\mathbb{Z}$: both are Euclidean Domains.

Euclidean Domains

Definition. A Euclidean Domain is an algebra $(R, +, \cdot, \mathrm{quot}, \mathrm{rem}, \mathrm{sgn})$ together with $\delta : R \setminus \{0\} \to W$ (W well-ordered) such that:

- $(R, +, \cdot)$ is an integral domain,
- $b = \mathrm{quot}(b, a) \cdot a + \mathrm{rem}(b, a)$,
- $a \neq 0 \implies \big(\mathrm{rem}(b, a) = 0 \ \vee \ \delta(\mathrm{rem}(b, a)) < \delta(a)\big)$,
- $\mathrm{rem}(b + ta, a) = \mathrm{rem}(b, a)$,
- $r\,\mathrm{sgn}(r)$ and r are associated (unit multiples) in R,
- r, s associated $\Rightarrow r\,\mathrm{sgn}(r) = s\,\mathrm{sgn}(s)$.

r is normalized if $r\,\mathrm{sgn}(r) = r$.

All Euclidean domains are PIDs. $\mathbb{Z}[\frac{1+\sqrt{19}\,i}{2}]$ is a PID, but not Euclidean.

Checking Similarity

Theorem. Let $A, B \in \mathbb{K}^{n\times n}$. The following are equivalent:

1. A and B are similar over $\mathbb{K}$.
2. $\mathbb{K}[t]^n / \operatorname{col}(t * I - A) \cong_{\mathbb{K}[t]} \mathbb{K}[t]^n / \operatorname{col}(t * I - B)$.
3. The Smith Normal Forms of $t * I - A$ and $t * I - B$ over $\mathbb{K}[t]$ are the same.

Checking Similarity

Let $A := \begin{pmatrix} 41 & -81 \\ 16 & -31 \end{pmatrix}$ and $B := \begin{pmatrix} -9 & -49 \\ 4 & 19 \end{pmatrix}$. Then

$$\underbrace{\begin{pmatrix} \frac{1}{81} & 0 \\ t+31 & -81 \end{pmatrix}}_{U_A} \cdot \begin{pmatrix} t-41 & 81 \\ -16 & t+31 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & \frac{41}{81} - \frac{t}{81} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (t-5)^2 \end{pmatrix}$$

and

$$\underbrace{\begin{pmatrix} \frac{1}{49} & 0 \\ t-19 & -49 \end{pmatrix}}_{U_B} \cdot \begin{pmatrix} t+9 & 49 \\ -4 & t-19 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & -\frac{t}{49} - \frac{9}{49} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (t-5)^2 \end{pmatrix}.$$

Let $U := U_B^{-1} U_A$ and $P[i,j] := \sum_{k=1}^{n} u_{\hat{k},j}(B)[i,k]$. Then $P := \begin{pmatrix} 1 & 0 \\ -\frac{50}{49} & \frac{81}{49} \end{pmatrix}$ satisfies $PAP^{-1} = B$.

Consequences

Suppose:

$$t * I - A \quad \text{is unimodular equivalent to} \quad S = \begin{pmatrix} d_1(t) & 0 & 0 & \dots & 0 \\ 0 & d_2(t) & 0 & \dots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & d_n(t) \end{pmatrix}.$$

Corollary I. V_A is isomorphic to $\prod_{i=1}^n \mathbb{K}[t]/(d_i)$ with $d_1 \mid d_2 \mid \cdots \mid d_n$ and

- $\det(t * I - A) = d_1 \cdots d_n$,
- hence $c_A = \det(t * I - A)$ annihilates $\prod_{i=1}^n \mathbb{K}[t]/(d_i)$ and thus $\hat{c_A}(A) \cdot v = 0$ for all $v \in \mathbb{K}^n$ (Cayley-Hamilton), and
- d_n is the minimal polynomial of A, and $c_A \mid {d_n}^n$.

Corollary II. If $\mathbb{K} \leq \mathbb{L}$ and $A, B \in \mathbb{K}^{n \times n}$ are similar over $\mathbb{L}$, then they are similar over $\mathbb{K}$.

Jordan Normal Form

Jordan Normal Form

Theorem (C. Jordan 1870). Let $A \in \mathbb{K}^{n\times n}$ be such that c_A splits into linear factors over $\mathbb{K}$. Then A is similar over $\mathbb{K}$ to

$$\begin{pmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & J_k \end{pmatrix} \text{ with } J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}.$$

Proof = Construction of a Jordan basis:

- $V_A \cong \prod \mathbb{K}[t]/(d_i) \cong \prod_{(i,j)\in I} \mathbb{K}[t]/(t-\lambda_{i,j})^{e_{i,j}} := W$.
- A acts on $\mathbb{K}^n$ like t on W. Formally: $\Phi(t \star_A [v]) = A \cdot \Phi([v])$ for the isomorphism $\Phi : K[t]^n/\operatorname{col}(t * I - A) \to \mathbb{K}^n$.
- For $\mathbb{K}[t]/(t-\lambda)^e$, take the basis $(b_k)_{k=0}^{e-1} = ([(t-\lambda)^{e-k}])_{k=0}^{e-1}$. Then $t \star b_k = (t-\lambda)b_k + \lambda b_k = b_{k-1} + \lambda b_k$.

Literature

- D. S. Dummit and R. M. Foote. *Abstract algebra.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.