A COMBINATORIAL CONSEQUENCE OF THE BAKER-PIXLEY-THEOREM ON SUBALGEBRAS OF DIRECT PRODUCTS

Erhard Aichinger and Bernardo Rossi
Institute for Algebra
Austrian Science Fund FWF P33878

JOHANNES KEPLER
UNIVERSITY LINZ

Subset systems

Let A be a set. $\mathcal{A}$ is a subset system if $\mathcal{A} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}(A^n)$.

Examples:

- **Algebraic sets** $A := \mathbb{R}$, $\mathcal{A} =$ all algebraic subsets of $\mathbb{R}^n$ with $n \in \mathbb{N}$.
- **Subalgebras** $A := S_3$, $\mathcal{A} =$ all subgroups of S_3^n with $n \in \mathbb{N}$.
- **Topologies** $A := \mathbb{R}$, $\mathcal{A} =$ all open subsets of $\mathbb{R}^n$ with $n \in \mathbb{N}$.

Closure properties of subset systems

$\mathcal{A}$ is closed under unions if

$$\forall n \in \mathbb{N}\ \forall X, Y \in A^n \quad X \in \mathcal{A} \text{ and } Y \in \mathcal{A} \Longrightarrow X \cup Y \in \mathcal{A}.$$

$\mathcal{A}$ is is closed under minors if for all $m, n \in \mathbb{N}$, $X \subseteq A^n$ with $X \in \mathcal{A}$, we have

$$\{(x_1, \ldots, x_m) \mid (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in X\} \in \mathcal{A}.$$

Description of subset systems

Theorem (EA and Rossi, 2020)

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subset systems on the finite set A. We assume that both systems are closed under minors, unions and intersections. Then $\mathcal{A}_1 = \mathcal{A}_2$ if and only if they contain the same subsets of $A^{|A^2|}$.

Baker-Pixley Theorem

Theorem (Baker, Pixley, 1975)

Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be algebras in a variety $\mathcal{V}$ that has a term m such that

$$\mathcal{V} \models m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Let $\mathbf{B}$ and $\mathbf{C}$ be subalgebras of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ such that for all $i, j \in \{1, \ldots, n\}$ we have

$$\{(a_i, a_j) \mid (a_1, \ldots, a_n) \in \mathbf{B}\} = \{(a_i, a_j) \mid (a_1, \ldots, a_n) \in \mathbf{C}\}.$$

Then $\mathbf{B} = \mathbf{C}$.

Example: In a lattice, $m(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is a majority term.

Consequences of the Baker-Pixley Theorem

Let $\mathbf{A}$ be an algebra with majority term, and let X be a finite set with $|X| \geq 2$, and let $\mathbf{F}$ be a subalgebra of $\mathbf{A}^X$.

If a function $g : X \to A$ can be interpolated at every two element subset by a function in $\mathbf{F}$, then $g \in \mathbf{F}$.

On the two element lattice $\{0, 1\}$, every n-ary monotonic function can be expressed by a term in $x_1, \ldots, x_n$ that uses $\wedge, \vee, 0, 1$ (but not $\neg$).

Towards more sophisticated consequences

The term functions of an algebra $\mathbf{A} = (A, f_1, f_2, \ldots)$ are those that can be expressed by terms.

Example $t(x_1, x_2, x_3) := x_1 \wedge (x_1 \vee x_3)$ in a lattice.

Properties of term functions on lattices:

- If $f(x_1, \ldots, x_m)$ and $g(x_1, \ldots, x_m)$ are term functions, then so are $h_1(x_1, \ldots, x_n) := f(x_1, \ldots, x_n) \wedge g(x_1, \ldots, x_n)$ and $h_2 := f \vee g$.
- If $f(x_1, \ldots, x_m)$ is a term function, then so is $h(x_1, \ldots, x_m) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, for every $\sigma : [n] \to [m]$.
  Example: $h(x_1, x_2) := f(x_1, x_1, x_1)$.
  We call h a minor of f.

Sparks' Lemma: Baker-Pixley for clonoids

Theorem (Sparks, 2019)

Let A be a finite set, and let $C \subseteq \{f \mid f : A^n \to \{0,1\}, n \in \mathbb{N}\}$.

We assume that C is closed under $\vee$, $\wedge$ and minors. (C is a clonoid.)

Let $g : A^N \to \{0,1\}$, and let $n := |A|^2$. Suppose that every n-ary minor of g is in C.

Then $g \in C$.

Theorem (Sparks, 2019)

Let A be a finite set, let $\mathbf{B} := (\{0,1\}, \wedge, \vee)$ be the two element lattice, and let C, D be two clonoids from A to $\mathbf{B}$. Then $C = D$ if and only if $C^{[\,|A|^2\,]} = D^{[\,|A|^2\,]}$.

Sparks' Lemma: Baker-Pixley for clonoids

Idea of the proof: By Baker-Pixley, it is sufficient to interpolate g at every two-element subset of A^N.

$A = \{a_1, \dots, a_m\}$.

$$\begin{aligned} g(a_1, a_2, a_1, a_3, a_1, \dots, a_1) &= z_1 \\ g(a_2, a_1, a_1, a_1, a_1, \dots, a_1) &= z_2 \end{aligned}$$

Instead of $g(x_1, \dots, x_N)$, interpolate

$$g(y_{1,2},\ y_{2,1},\ y_{1,1},\ y_{3,1},\ y_{1,1},\ \dots,\ y_{1,1})$$

in m^2 variables.

The subset system theorem

Theorem (EA and Rossi, 2020)

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subset systems on the finite set A. We assume that both systems are closed under minors, unions and intersections. Then $\mathcal{A}_1 = \mathcal{A}_2$ if and only if they contain the same subsets of $A^{|A^2|}$.

Proof: Replace the subsets by their characteristic functions.

The set of characteristic functions is closed under $\wedge, \vee$ and minors. Then apply Sparks' Lemma.

A logical point of view

Definition

Let $\mathbf{A} = (A, (f_i)_{i \in I}, (\rho_j)_{j \in J})$ be a first order structure, let Φ be a set of first order formulas in the language of $\mathbf{A}$, let $n \in \mathbb{N}$, and let $B \subseteq A^n$. B is Φ-definable if there is a formula $\varphi \in \Phi$ whose free variables are all contained in $\{x_1, \ldots, x_n\}$ such that

$$B = \{(a_1, \ldots, a_n) \in A^n \mid \mathbf{A} \models \varphi(a_1, \ldots, a_n)\}.$$

$\mathrm{Def}^{[n]}(\mathbf{A}, \Phi)$ denotes the set of all Φ-definable subsets of A^n, $\mathrm{Def}(\mathbf{A}, \Phi) := \bigcup_{n \in \mathbb{N}} \mathrm{Def}^{[n]}(\mathbf{A}, \Phi)$.

Minors of formulae

A first order formula φ is a minor of the formula φ' if $\exists n \in \mathbb{N}, \sigma : \{1, \ldots, n\} \to \mathbb{N}$ such that

$$\varphi = \varphi' \frac{x_{\sigma(1)}, \ldots, x_{\sigma(n)}}{x_1, \ldots, x_n}.$$

Sometimes we write $\varphi = \varphi'(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Closure properties of definable sets

Let $\mathbf{A}$ be a first order structure, and let Φ be a set of first order formulas in its language closed under $\wedge$, $\vee$, and taking minors of formulas. Then $\mathrm{Def}(\mathbf{A}, \Phi)$ is closed under finite intersections, finite unions, and taking minors of sets.

The subset system theorem phrased in logical terms

Theorem (EA, Rossi, 2020)

Let $\mathbf{A}$ be a finite fo structure, let Φ be a set of fo formulas closed under $\wedge$, $\vee$, and minors. Let $m := |A|^2$, and $(S_i)_{i \in I}$ be a family of subsets of A^m such that

$$\{S_i \mid i \in I\} = \mathrm{Def}^{[m]}(\mathbf{A}, \Phi).$$

Let $(\sigma_i)_{i \in I}$ be a family of relational symbols of arity m. For each $i \in I$, let $\sigma_i^{\mathbf{A}'} := S_i$, and let $\mathbf{A}'$ be the relational structure $(A, (\sigma_i^{\mathbf{A}'})_{i \in I})$.

Let Ψ be the closure of $\{\sigma_i(x_1, \dots, x_m) \mid i \in I\}$ under minors, $\wedge$, and $\vee$. Then

$$\mathrm{Def}(\mathbf{A}, \Phi) = \mathrm{Def}(\mathbf{A}', \Psi).$$

Equational Domains

Two algebras are algebraically equivalent if they have the same algebraic geometry.

An algebra is an equational domain if the finite union of algebraic sets is algebraic.

Theorem (Pinus 2017)

Let A be a finite set, and let $(\mathbf{A}_i)_{i \in I}$ be a family of pairwise algebraically inequivalent equational domains with universe A. Then I is finite.

Closure Properties of Subalgebras

Let $\mathbf{A} = (A, F)$ be an algebra.

$$\mathcal{S}(\mathbf{A}) := \bigcup_{n\in\mathbb{N}} \{\mathbf{B} \mid \mathbf{B} \leq \mathbf{A}^n, n \in \mathbb{N}\}.$$

$\mathcal{S}(\mathbf{A})$ is closed under projections and products. It is called the system of subpowers of $\mathbf{A}$.

Y is a projection of $X \subseteq \mathbf{A}^m$ if $Y = \{(x_{\alpha(1)}, \dots, x_{\alpha(n)}) \mid (x_1, \dots, x_m) \in X\}$.

Z is the product of $X \subseteq A^m$ and $Y \subseteq A^n$ if
$Z = \{(x_1, \dots, x_m, y_1, \dots, y_n) \mid (x_1, \dots, x_m) \in X, (y_1, \dots, y_n) \in Y\}$.

Closure Properties

	$\cap$	$\cup$	projections	minors	products
subalgebras	$\bullet$		$\bullet$	$\bullet$	$\bullet$
algebraic sets	$\bullet$			$\bullet$	$\bullet$

Closure Properties

Theorem (Pöschel, Kalužnin 1979)

Let A be a finite set. If the subset system $\mathcal{S}$ on A is closed under projections, minors, and products, then there is an algebra (A, F) such that $\mathcal{S}$ is the system of subpowers of (A, F).

Corollary

A subset system $\mathcal{S}$ on a finite set is a system of subpowers if and only if it closed under primitive positive definitions.

Primitive positive definitions

A primitive positive formula $\varphi(x_1, \ldots, x_n)$ is a f.o. formula of the type

$$\exists x_{i_1}, \ldots, x_{i_m} \psi_1 \wedge \ldots \wedge \psi_k,$$

where all ψ_i are atomic formulae.

If $S \subseteq A^3$ and $T \subseteq A^4$, then
$X = \{(a_1, a_2) \mid \exists a_3, a_4 : (a_1, a_3, a_1) \in S \wedge (a_2, a_4, a_4, a_2) \in T\}$ is definable from S, T by primitive positive definition.

A finite basis theorem

Theorem (EA, Mayr, McKenzie 2014)

Let $\mathbf{A}$ be a finite algebra with a Mal'cev term ($m(x,x,y) = m(y,x,x) = y$). Then there is $k \in \mathbb{N}$ and $B \subseteq A^k$ such that every subpower is definable from B using primitive positive definitions.

Corollary

Let G be a finite group, $|G| > 1$. Then there exists $k \in \mathbb{N}$ and $H \leq G^k$ such that for every $n \in \mathbb{N}$, $S \leq G^n$, there are $l, m \in \mathbb{N}$, $\sigma : \underline{m} \times \underline{k} \to \underline{l}$, $\tau : \underline{n} \to \underline{l}$ such that

$$\begin{aligned} S = \{ \quad & (g_1, \ldots, g_n) \in G^n \,\big|\, \exists a_1, \ldots, a_l \in G : \\ & \textstyle\bigwedge_{i \in \underline{m}} (a_{\sigma(i,1)}, \ldots, a_{\sigma(i,k)}) \in H \\ & \wedge \\ & g_1 = a_{\tau(1)} \wedge \ldots \wedge g_n = a_{\tau(n)} \}. \end{aligned}$$

A consequence on groups

Theorem

Let G be a finite group. Then there are $k \in \mathbb{N}$, $H \leq G^k$ such that $\mathcal{S} := \bigcup_{n\in\mathbb{N}} \mathrm{Sub}(G^n)$ is the smallest set such that

- $H \in \mathcal{S}$;
- $\forall m, n \in \mathbb{N}, A \in \mathcal{S}^{[m]}, \sigma : \underline{n} \to \underline{m}$ we have $\{(h_{\sigma(1)}, \ldots, h_{\sigma(n)}) \,|\, (h_1, \ldots, h_m) \in A\} \in \mathcal{S}^{[n]}$;
- $\forall m, n \in \mathbb{N}, A \in \mathcal{S}^{[n]}, \sigma : \underline{n} \to \underline{m}$ we have $\{(h_1, \ldots, h_m) \,|\, (h_{\sigma(1)}, \ldots, h_{\sigma(n)}) \in A\} \in \mathcal{S}^{[m]}$;
- $\forall n \in \mathbb{N}, A, B \in \mathcal{S}^{[n]} : A \cap B \in \mathcal{S}^{[n]}$.

Theorem TBG

Let A be a set. If the subset system $\mathcal{S}$ on A is closed under projections, minors, and products, then $\mathcal{S}$ is the system of subpowers of some algebra.

Questions:

- What are systems that are closed under minors and products?
- What are the closure properties characterizing algebraic geometries?