Clonoids

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Outline

Definition of Clonoids

Setup:

A	$\dots$	set
$\mathbf{B}$	$\dots$	algebra
C	$\dots$	finitary functions from A to B, hence
C	$\subseteq$	$\bigcup_{n \in \mathbb{N}} B^{A^n}$
$C^{[k]}$	$:=$	$C \cap B^{A^k}$ (k-ary functions in C).

$C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}$ is a

clonoid with source set A and target algebra $\mathbf{B}$ $:\Longleftrightarrow$

- for all $k \in \mathbb{N}$: $C^{[k]}$ is a subuniverse of $\mathbf{B}^{A^k}$, and
- for all $k, n \in \mathbb{N}$, for all $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$, and for all $c \in C^{[k]}$, the function $c' : A^n \to B$ defined by

$$c'(a_1, \dots, a_n) := c(a_{i_1}, \dots, a_{i_k})$$

satisfies $c' \in C^{[n]}$.

Definition as stable classes

Composition of function classes

[Couceiro and Foldes, Acta Cybernetica 2007]

$\mathrm{Fin}(A, B) := \bigcup_{n\in\mathbb{N}} B^{A^n}$ = all finitary functions from A to B

$X \subseteq \mathrm{Fin}(A, B),\ Y \subseteq \mathrm{Fin}(B, C)$

$YX := \{g(f_1, \dots, f_n) \mid m, n \in \mathbb{N}, g \in Y^{[n]}, f_1, \dots, f_n \in X^{[m]}\}$.

Associativity Lemma [Couceiro and Foldes]

Let $J := \{\pi_i^{(n)} : (x_1, \dots, x_n) \mapsto x_i \mid n, i \in \mathbb{N}, i \leq n\}$.
Then $(XY)Z \subseteq X(YZ) \subseteq (X(YJ))Z$.

Names of closed sets of finitary functions

$X \subseteq \mathrm{Fin}(A, A)$ is a **clone** $\iff J \subseteq X$, $XX \subseteq X$.

- $\{(a_1, \ldots, a_n) \mapsto p(a_1, \ldots, a_n) \mid n \in \mathbb{N}, p \in \mathbb{Z}[x_1, \ldots, x_n]\}$ is a clone on $\mathbb{Z}$.
- For every universal algebra $\mathbf{A} = (A, f_1, \ldots, f_k)$, the set of **term functions** of $\mathbf{A}$ is a clone on A denoted by $\mathrm{Clo}(\mathbf{A})$.
- $J_A := \{\pi_i^{[n]}(x_1, \ldots, x_n) \mapsto x_i \mid n, i \in \mathbb{N}, i \leq n\}$ is the smallest clone on the set A and called the **clone of projections.**

Names of closed sets of finitary functions

Let C be a clone on A, and let D be a clone on B.
$X \subseteq \mathrm{Fin}(A, B)$ is

- (C, D)**-stable** [Couceiro and Foldes 2009] $\iff XC \subseteq X$ and $DX \subseteq X$.
- **minor closed set** or **minion** $\iff XJ_A \subseteq X \iff X$ is (J_A, J_B)-stable.
- If $\mathbf{B}$ is an algebra, then X is a **clonoid with source set** A **and target algebra** $\mathbf{B} \iff X$ is $(J_A, \mathrm{Clo}(\mathbf{B}))$-stable.

Basic Theory of Clonoids

- $X \subseteq \mathrm{Fin}(A, B)$ is a clonoid from A into $\mathbf{B} = (B; G)$
  $\iff$ there exists a two-sorted algebra $\mathbf{C} = ((A, B); F \cup G)$ with $F \subseteq \mathrm{Fin}(A, B)$ and $G \subseteq \mathrm{Fin}(B, B)$ such that

  X is the set of (finitary) **term functions** from A to B.

- Let $R \subseteq A^I$, $S \subseteq B^I$, X clonoid from A to $(B; G)$.
  The pair (R, S) is **preserved** by X
  $\iff$ (R, S) is a subuniverse of $((A, B); X)^I \iff XR \subseteq S$.
  Then (R, S) is an **invariant relation** of X.

The "Polym-Inv"-Theorem

Let A be finite, $f : A^n \to B$, X a minion from A to B. If f preserves

$$(\{\pi_i^{(n)} \mid i \in \{1, \ldots, n\}\}, \; X^{[n]}) \in P(A^{A^n}) \times P(B^{A^n}),$$

then $f \in X$.

Hence each clonoid with finite source is determined by its finitary invariant relation pairs.

The "Inv-Polym"-Theorem [Pippenger 2002]

Let A, B be finite, $R \subseteq A^k$, $S \subseteq B^k$. Let

$$X = \{f : A^n \to B \mid n \in \mathbb{N},\ f \text{ preserves } (R, S)\}.$$

If $(C, D) \leq ((A, B), X)^m$, then (C, D) can be obtained from (R, S) using

- **direct products**
- **minors**: $\exists \sigma$ with $\begin{cases} C = \{(c_1, \ldots, c_m) \in A^m \mid (c_{\sigma(1)}, \ldots, c_{\sigma(k)}) \in R\} \\ D = \{(d_1, \ldots, d_m) \in B^m \mid (d_{\sigma(1)}, \ldots, d_{\sigma(k)}) \in S\} \end{cases}$
- **projections**: $\exists \sigma$ with $\begin{cases} C = \{(c_{\sigma(1)}, \ldots, c_{\sigma(m)}) \mid (c_1, \ldots, c_k) \in R\} \\ D = \{(d_{\sigma(1)}, \ldots, d_{\sigma(m)}) \mid (d_1, \ldots, d_k) \in S\} \end{cases}$
- **relaxations**: If $f : A^n \to B$ preserves (R, S), then it also preserves (R', S') with $R' \subseteq R$, $S \subseteq S'$.

Occurrences of clonoids

Clonoids inside clones

Given: an algebra $\mathbf{A}$.
Asked: describe the polynomial functions of $\mathbf{A}$.

Example: On a finite field, every function is a polynomial function.
On the lattice $(\{0, 1\}, \vee, \wedge)$, every monotonic function is polynomial.
In describing $\mathrm{Pol}(\mathbf{A}) \subseteq \mathrm{Fin}(A, A)$, one oftens describes **clonoids** inside $\mathrm{Pol}(\mathbf{A})$.

Clonoids inside clones

Let $\mathbf{A}$ be a universal algebra, and let $\rho \in \mathrm{Con}(\mathbf{A})$, $o \in A$.

- $P_0 =$ those functions that fix o = $\mathrm{Polym}(\{o\}, \{o\})$.
- All functions that map into o/ρ = $\mathrm{Polym}(A, o/\rho)$.
  This is a $(\mathrm{Pol}(\mathbf{A}), P_0)$-stable class.
  For submodules I, J of an R-module M, the *Noetherian quotient* $(I : J) = \{r \in R \mid rJ \subseteq I\}$ "is" $\mathrm{Polym}(J, I) \cap \mathrm{Clo}^{[1]}({}_R M)$.
- All functions that are constant on ρ-cosets = $\mathrm{Polym}(\rho, =_A)$.
  This is a $(\mathrm{Pol}(\mathbf{A}), \mathrm{Pol}(\mathbf{A}))$-stable class.
- All functions that map into one ρ-class = $\mathrm{Polym}(A \times A, \rho)$. This is a $(\mathrm{Pol}(\mathbf{A}), \mathrm{Pol}(\mathbf{A}))$-stable class.

For example, [EA, 2006] describes the polynomial functions of the finite nonsolvable special linear groups.

Occurrences in clone theory

We let A, B be abelian groups, $f : A^n \to B$.

- For $a \in A^n$, $\Delta_a(f)\,(x) := f(x + a) - f(x)$.
- $\textsc{Fdeg}(f) :=$ the minimal $k \in \mathbb{N}_0$ with $\Delta_{a_1}\Delta_{a_2}\cdots\Delta_{a_{k+1}} f = 0$ for all $a_1, \ldots, a_{k+1} \in A^n$.
- **Intuitive:** $f : \mathbb{R}^1 \to \mathbb{R}$ is a polynomial of degree $\leq 2 \iff f''' = 0$.

By [Leibman 2002], we have

Lemma

Let $\mathrm{End}(A)$ be the set of all endomorphims from A^n to A, $n \in \mathbb{N}$.
Then for $k \in \mathbb{N}_0$,

$$D_k := \{f : A^n \to B \,|\, \textsc{Fdeg}(f) \leq k\}$$

is an $(\mathrm{End}(A), \mathrm{End}(B))$-stable class.

Clonoids in universal algebraic geometry

Let $S \subseteq \bigcup_{n\in\mathbb{N}} P(A^n)$ be a system of finitary relations on A. Then:

- If for all $n, k, \sigma : [n] \to [k]$, $B \in S$ with $B \subseteq A^n$,

  $$B_\sigma := \{(s_1, \ldots, s_k) \in A^k \mid (s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \in B\} = \{s \in A^k \,|\, s \circ \sigma \in B\}$$

  satisfies $B_\sigma \in S$, then the set of characteristic functions

  $$C_S := \{\chi_B \mid B \in S\}$$

  is a clonoid from A into $\{0, 1\}$ called **the characteristic clonoid** of S. B_σ is a **minor** of B.
- If S is closed under intersections, C_S is a clonoid from A into $(\{0, 1\}, \wedge)$.
- If S is closed under intersections and unions, C_S is a clonoid from A into $(\{0, 1\}, \wedge, \vee)$.

Minor closed subsets in complexity theory

We look for an algorithm that does the following:

Input: A finite graph $\mathbb{G}$.
Output: **Yes** if $\mathbb{G}$ is 3-colorable. **No** if $\mathbb{G}$ is not even 100-colorable.
Do not bother about boarderline cases (**G** is 100-colorable and not 3-colorable).
This is PCSP$(\mathbb{K}_3, \mathbb{K}_{100})$, where $\mathbb{K}_n = ([n], \neq)$.

Theorem [Pippenger 2002; Brakensiek, Guruswami 2019; Barto, Bulín, Krokhin, Opršal 2019]

If $\mathrm{Polym}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{Polym}(\mathbb{A}', \mathbb{B}')$, then PCSP$(\mathbb{A}', \mathbb{B}')$ is polynomial-time reducible to PCSP$(\mathbb{A}, \mathbb{B})$.

Clonoids in equational logic

The equational theory of W in $\mathbf{A}$ [EA and Mayr 2016]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

$$s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A} \text{ with } W \models s \approx t\}.$$

$\mathrm{Th}_{\mathbf{A}}(W)$ is a clonoid with source set A and target algebra $\mathbf{A} \times \mathbf{A}$.

These clonoids from A into the algebra $\mathbf{A} \times \mathbf{A}$ can distinguish all subvarieties of $\mathbb{V}(\mathbf{A})$.

The finite relatedness result

Finite relatedness of Mal'cev clonoids

Theorem [EA Mayr McKenzie 2014, 2016]

Let A be finite, **B** be a finite algebra with a Mal'cev term, and let C be a clonoid from A to **B**. Then

1. There are $n \in \mathbb{N}$, $R \subseteq A^n$, $S \leq \mathbf{B}^n$ such that $C = \mathrm{Polym}(R, S)$.
2. There is no infinite descending chain of clonoids from A to **B**,

Corollary

Every subvariety of a finitely generated variety with Mal'cev term is finitely generated.

We will now explain the main steps in the proof.

Forks

Definition of Forks

Let $\mathbf{A}$ be an algebra, let $m \in \mathbb{N}$, and let F be a subuniverse of $\mathbf{A}^m$. For $i \in \{1, \ldots, m\}$, we define the relation $\mathrm{Forks}_i(F)$ on A by

$$\mathrm{Forks}_i(F) := \{(a_i, b_i) \mid (a_1, \ldots, a_m) \in F, (b_1, \ldots, b_m) \in F, \\ (a_1, \ldots, a_{i-1}) = (b_1, \ldots, b_{i-1})\}.$$

If $(c, d) \in \mathrm{Forks}_i(F)$, we call (c, d) a fork of F at i.

If

$$\begin{aligned} \mathbf{u} &= (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m) \in F \text{ and} \\ \mathbf{v} &= (a_1, \ldots, a_{i-1}, d, b_{i+1}, \ldots, b_m) \in F, \end{aligned}$$

then $(\mathbf{u}, \mathbf{v})$ is a witness of the fork (c, d) at i.

Forks in linear algebra

Let $\mathbf{F} \leq (\mathbb{Z}_5, +)^4$ with $\mathbf{F} = \langle (1, 2, 2, 2), (3, 1, 4, 2), (1, 2, 0, 3) \rangle$. When we compute the row echelon form of

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 3 & 1 & 4 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we obtain that $(0, 1)$ is a fork of $\mathbf{F}$ at 1 with witness $(0, 0, 0, 0), (1, 2, 0, 3)$ and that $(0, 1)$ is a fork of $\mathbf{F}$ at 3 with witness $(0, 0, 0, 0), (0, 0, 1, 2)$.

Significance of forks

Let $\mathbf{A}$ be an algebra with a Mal'cev term, and let $\mathbf{F} \leq \mathbf{A}^m$.

- Let $G \subseteq F$ be such that G contains at least one witness for each fork of $\mathbf{F}$. Then G generates $\mathbf{F}$.

  Hence for $a := |A| < \aleph_0$, $\mathbf{F}$ can be generated by $a^2 \cdot m$ elements and thus

$$\#\,\mathrm{Sub}(\mathbf{A}^m) \leq \binom{a^m}{a^2 m} \leq a^{m \cdot a^2 \cdot m} \leq 2^{C(a) m^2}.$$

- (The **Fork Lemma**)
  If $\mathbf{E} \leq \mathbf{F}$ and every fork of $\mathbf{F}$ is a fork of $\mathbf{E}$, then $\mathbf{E} = \mathbf{F}$.

Representation of Clonoids

We represent a clonoid C with source set $A = \{a_1, \ldots, a_t\}$ and target algebra $\mathbf{B}$ using forks.

For $\mathbf{a} \in A^n$, we define the forks of C at $\mathbf{a}$ by

$$\mathrm{Forks}(C, \mathbf{a}) := \{\big(f_1(\mathbf{a}), f_2(\mathbf{a})\big) \in B \times B \mid f_1 \in C, f_2 \in C, \\ f_1(\mathbf{z}) = f_2(\mathbf{z}) \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\text{lex}} \mathbf{a}\}.$$

Fork Lemma for Clonoids

A finite set, $\mathbf{B}$ finite algebra with Mal'cev term, C, D clonoids from A to $\mathbf{B}$. If $C \subseteq D$ and C and D have the same forks, then $C = D$.

Connections between forks of different arity

$\mathbf{a} \leq_E \mathbf{b} :\Leftrightarrow \mathbf{b}$ can be obtained from $\mathbf{a}$ by inserting additional letters anywhere after their first occurrence in $\mathbf{a}$. Such orders are studied further in [McDevitt 2018].

$$\text{abac} \leq_E \text{aaababbaabaccba}$$

Embedded Forks Lemma ([EA Mayr McKenzie 2014] for clones)

Let A be a set and $\mathbf{B}$ be an algebra, let C be a clonoid from A to $\mathbf{B}$, and let $\mathbf{a}, \mathbf{b} \in A^*$ with $\mathbf{a} \leq_E \mathbf{b}$.
Then $\mathrm{Forks}(C, \mathbf{b}) \subseteq \mathrm{Forks}(C, \mathbf{a})$.

Facts on the embedding orderings

For a finite set A, $(A^*, \leq_E)$ has no infinite descending chains and no infinite antichains. In other words, $(A^*, \leq_E)$ is **well partially ordered.**
Let U be the set of upward closed subsets of $(A^*, \leq_E)$. Then

- $(U, \subseteq)$ has no infinite ascending chains.
  Reason: $\bigcup_{i \in \mathbb{N}} U_i$ is generated by its minimal elements $M \subseteq A^*$. As an antichain in $(A^*, \leq_E)$, M is finite.
  Hence there is $j \in \mathbb{N}$ with $M \subseteq U_j$, und thus $U_{j+1} \subseteq U_j$.
- $(U, \subseteq)$ has no infinite antichains. Well-known; a proof in [EA and Aichinger, Expo. Math. (2020)].

Proof of the finite relatedness result

Theorem [EA Mayr McKenzie 2014, 2016]

Let A be finite, $\mathbf{B}$ be a finite algebra with a Mal'cev term. Then there is no infinite descending chain of clonoids from A to $\mathbf{B}$.

Proof (sketch):

- Let $(C_i)_{i \in \mathbb{N}}$ be an infinite descending chain of clonoids from A to $\mathbf{B}$.
- For each $\mathbf{D} \leq \mathbf{B} \times \mathbf{B}$, $U_{\mathbf{D}}(C_i) := \{\mathbf{a} \in A^* \mid \mathrm{Forks}(C_i, \mathbf{a}) \subseteq D\}$ is upward closed by the Embedded Forks Lemma.
- There is $\mathbf{D} \subseteq \mathbf{B} \times \mathbf{B}$ such that $(U_{\mathbf{D}}(C_i))_{i \in \mathbb{N}}$ is an infinite ascending chain. Contradiction.

A "constructive" version

Theorem [EA Mayr McKenzie 2014, 2016]

Let A be finite, $\mathbf{B}$ be a finite algebra with a Mal'cev term, and let C be a clonoid from A to B. Let

$$m := \max\{|\mathbf{a}| \mid \exists \mathbf{D} \leq \mathbf{B} \times \mathbf{B} : \mathbf{a} \text{ is minimal in } U_{\mathbf{D}}(C)\}.$$

Then $C = \mathrm{Polym}(R; S)$ with $R \subseteq A^{|A|^m}$ and $S \leq \mathbf{B}^{|A|^m}$.

We do not know how to compute m. Computing m would allow us to decide:

Given: $F \subseteq_{\mathrm{fin}} \mathrm{Fin}(B, B)$, $k \in \mathbb{N}$, $\rho \subseteq B^k$.
Asked: $\mathrm{Clo}_B(F \cup \{d\}) = \mathrm{Polym}(\rho)$.

Note that $\subseteq$ is easy to check.

Open problems - clones and clonoids

1. Release finiteness of A or **B**.
2. A finite, **B** Mal'cev algebra. Is there an infinite antichain of clonoids from A to **B**?
3. Is there an infinite antichain of clones with a Mal'cev term on a finite set?

Significance of “no antichains”

We imitate [Robertson and Seymour, *Graph minors. XX.* 2004].

Fact

A finite, d Mal'cev, f, g operations on A.
Suppose that there is no infinite antichain of clones on A containing d. We define

$$f \leq_d g :\Longleftrightarrow f \in \mathrm{Clo}_A(g, d).$$

Let ψ be a property of operations such that

$$g \models \psi, f \leq_d g \Rightarrow f \models \psi.$$

Then ψ can be decided in polynomial time in $||f|| \sim |A|^{\mathrm{arity}(f)}$.

Proof: ψ has finitely many minimal counterexamples $g_1, \ldots, g_k$. The property $g_i \in \mathrm{Clo}_A(f, d)$ can be checked “easily”.

Open problems - varieties

The "no descending chains" result for clonoids yields:

Theorem [EA and Mayr 2016]

Let **A** be a finite algebra with a Mal'cev term. Then every subvariety of $\mathbb{V}(\mathbf{A})$ is generated by a finite algebra.

Open problems:

- Given $\mathbf{A}, \mathbf{B}$ similar finite algebras with Mal'cev term, $\mathbb{V}(\mathbf{A}) \cap \mathbb{V}(\mathbf{B})$ is therefore finitely generated. Can you give an upper bound for the size of a generator?
- Is there a finitely generated variety with a Mal'cev term (of finite type) with an infinite antichain of subvarieties? (Infinite ascending chains cannot exist, and infinite descending chains sometimes do exist; pointed group [Bryant 1982]).

Description of concrete clonoids

Clonoids closed under near-unanimity terms

$t(x_1, \dots, x_n)$ is NU-term $\iff$
$t(x, y, y, \dots, y) = t(y, x, y, \dots, y) = \cdots = t(y, y, \dots, x) = y$ for all x, y.

Theorem [Baker and Pixley 1975; Sparks 2019]

A finite, $\mathbf{B}$ algebra with n-ary NU-term Let C, D clonoids from A to $\mathbf{B}$ that have the same functions of arity $|A|^{n-1}$. Then $C = D$.

Remark: For clones, $|A|^{n-1}$ can be improved [Lakser, 1989], [Kerkhoff 2011], [Kerkhoff and Zhuk, 2014].

Clonoids closed under near-unanimity terms: a logical consequence

For a set of f.o. formulas Φ in the language of the f.o. structure **A**, let

$$\mathrm{Def}^{[n]}(\Phi) := \Big\{ \{(a_1, \ldots, a_n) \in A^n \mid \mathbf{A} \models \varphi(a_1, \ldots, a_n)\} \;\Big|\; \varphi \in \Phi \text{ with } \mathrm{freeVars}(\varphi) \subseteq \{x_1, \ldots, x_n\} \Big\}.$$

Corollary [EA and Rossi 2020]

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be f.o. structures on a finite set A. For each $i \in \{1, 2\}$, let Φ_i be a set of f.o. formulas in the language of $\mathbf{A}_i$ that is closed under $\wedge$, $\vee$, and substituting variables. Then $\mathrm{Def}(\mathbf{A}_1, \Phi_1) = \mathrm{Def}(\mathbf{A}_2, \Phi_2)$ if and only if $\mathrm{Def}^{[\,|A|^2\,]}(\mathbf{A}_1, \Phi_1) = \mathrm{Def}^{[\,|A|^2\,]}(\mathbf{A}_2, \Phi_2)$.

Clonoids closed under near-unanimity terms

By Sparks' Theorem, there are only finitely many clonoids from A into an algebra with NU-term.

Open problem [Sparks 2019]

Is there a finite set A and an algebra $\mathbf{B}$ such that

- there are only finitely many clonoids from A to $\mathbf{B}$ and
- $|A| > 1$ and $\mathbf{B}$ has no NU-term?

Clonoids with 2-element target

Theorem [Sparks 2019]

For $|\mathbf{B}| = 2$ and finite A, let c be the number of clonoids from A to $\mathbf{B}$. Then:

1. $c < \aleph_0 \iff \mathbf{B}$ has an NU-term.
2. $c = \aleph_0 \iff \mathbf{B}$ has a Mal'cev term and no NU-term.
3. $c = 2^{\aleph_0} \iff \mathbf{B}$ has neither NU nor Mal'cev term.

Linearly closed clonoids

Motivation: Bulatov and Idziak described all extensions of

$$\mathrm{Pol}(\mathbb{Z}_p \times \mathbb{Z}_p, +) = \mathrm{Clo}(\mathbb{Z}_p \times \mathbb{Z}_p, +, 1).$$

We try to describe extensions of $\mathrm{Clo}(\mathbb{Z}_p \times \mathbb{Z}_p, +)$.

Linear closed clonoids from $\mathbb{Z}_p$ to $\mathbb{Z}_p$

First step:

Theorem [Kreinecker 2020]

Let $L := \mathrm{Clo}(\mathbb{Z}_p, +)$. For every (L, L)-stable subclass C of $\mathrm{Fin}(\mathbb{Z}_p, Z_p)$, there is $M \subseteq \mathbb{N}_0$ such that

- $\forall n \in \mathbb{N} : (n \in M \wedge n > p - 1) \Rightarrow n - (p - 1) \in M$.
- $f \in C \iff f$ is induced by a polynomial such that the total degree of each monomial is in M.

The correspondence $C \mapsto M$ is a lattice isomorphism from the set (L, L)-stable classes to the subsets of $\mathbb{N}_0$ satisfying the first condition.

Example: $p = 3$, $M = \{6, 4, 2\} \cup \{11, 9, 7, 5, 3, 1\}$ is minor closed, $x_1 x_2 x_8 x_9 x_{10}^2 + x_1^2 x_2^2 - x_1 x_2 x_3 x_4 x_5 x_6 x_7 \in C$.
Hence the lattice of (L, L)-stable classes is isomorphic to $(\mathbb{N}_0 \cup \{\infty\}, \subseteq)^{p-1} \times$ 2-element chain.

Embedding the clonoid into a clone

Theorem [Fioravanti and Kreinecker 2020]

For an (L, L)-stable subclass C of $\mathrm{Fin}(\mathbb{Z}_p, \mathbb{Z}_p)$, define

$$\phi(C) := \{((x_1, y_1), \dots, (x_n, y_n)) \mapsto \Big(\sum_{i=1}^{n} a_i x_i + c(y_1, \dots, y_n), \sum_{i=1}^{n} b_i y_i\Big) \mid a_i, b_i \in \mathbb{Z}, c \in C\} \subseteq \mathrm{Fin}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p).$$

Then $\phi(C)$ is a clone. The mapping $C \mapsto \phi(C)$ is a lattice embedding.

Theorem [Kreinecker 2020]

Let $p > 2$ be a prime. Then there are infinitely many not finitely generated clones on $\mathbb{Z}_p \times \mathbb{Z}_p$ which contain $+$.

Extensions of $\mathrm{Clo}(\mathbb{Z}_p \times \mathbb{Z}_q, +)$

Theorem [Fioravanti 2020]

Let $s \in \mathbb{N}$ be squarefree. Then each clone containing $\mathrm{Clo}(\mathbb{Z}_s, +)$ is generated by its functions of arity $\leq s$.

Lemma [Fioravanti 2019]

Let $L_p := \mathrm{Clo}(\mathbb{Z}_p, +)$ and $L_q := \mathrm{Clo}(\mathbb{Z}_q, +)$. Then every (L_p, L_q)-stable subclass of $\mathrm{Fin}(\mathbb{Z}_p, \mathbb{Z}_q)$ is generated by one unary function.

¡Muchas gracias por su atención y esta invitación muy apreciada!