Clonoids

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Outline

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Definition of Clonoids

Setup:

\[ A \quad \ldots \quad \text{set} \]
\[ B \quad \ldots \quad \text{algebra} \]
\[ C \quad \ldots \quad \text{finitary functions from } A \text{ to } B, \text{ hence} \]
\[ C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n} \]
\[ C^{[k]} := C \cap B^{A^k} \quad (k\text{-ary functions in } C). \]

\( C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n} \) is a clonoid with source set \( A \) and target algebra \( B \) :

\[ \iff \]

\[ \text{for all } k \in \mathbb{N}: \ C^{[k]} \text{ is a subuniverse of } B^{A^k}, \text{ and} \]
\[ \text{for all } k, n \in \mathbb{N}, \text{ for all } (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k, \text{ and for all } c \in C^{[k]}, \text{ the function} \]
\[ c' : A^n \to B \text{ defined by} \]
\[ c'(a_1, \ldots, a_n) := c(a_{i_1}, \ldots, a_{i_k}) \]
\[ \text{satisfies } c' \in C^{[n]}. \]
Definition as stable classes

Composition of function classes

[Couceiro and Foldes, Acta Cybernetica 2007]

\[
\text{Fin}(A, B) := \bigcup_{n \in \mathbb{N}} B^A^n = \text{all finitary functions from } A \text{ to } B
\]

\[
X \subseteq \text{Fin}(A, B), \ Y \subseteq \text{Fin}(B, C)
\]

\[
YX := \{ g(f_1, \ldots, f_n) \mid m, n \in \mathbb{N}, g \in Y^n, f_1, \ldots, f_n \in X^m \}.
\]

Associativity Lemma [Couceiro and Foldes]

Let \( J := \{ \pi_i^{(n)} : (x_1, \ldots, x_n) \mapsto x_i \mid n, i \in \mathbb{N}, i \leq n \} \).

Then \((XY)Z \subseteq X(YZ) \subseteq (X(YJ))Z.\)
Names of closed sets of finitary functions

\( \mathcal{X} \subseteq \text{Fin}(A, A) \) is a clone \( \iff \mathcal{J} \subseteq \mathcal{X}, \mathcal{XX} \subseteq \mathcal{X} \).

\( \uparrow \quad \{(a_1, \ldots, a_n) \mapsto p(a_1, \ldots, a_n) \mid n \in \mathbb{N}, p \in \mathbb{Z}[x_1, \ldots, x_n] \} \) is a clone on \( \mathbb{Z} \).

\( \uparrow \quad \text{For every universal algebra } A = (A, f_1, \ldots, f_k), \text{ the set of term functions of } A \text{ is a clone on } A \text{ denoted by } \text{Clo}(A) \).

\( \uparrow \quad J_A := \{\pi_i^n(x_1, \ldots, x_n) \mapsto x_i \mid n, i \in \mathbb{N}, i \leq n\} \) is the smallest clone on the set \( A \) and called the clone of projections.
Let $C$ be a clone on $A$, and let $D$ be a clone on $B$. $X \subseteq \text{Fin}(A, B)$ is

- **$(C, D)$-stable** [Couceiro and Foldes 2009] $\iff XC \subseteq X$ and $DX \subseteq X$.
- minor closed set or minion $\iff XJ_A \subseteq X \iff X$ is $(J_A, J_B)$-stable.
- If $B$ is an algebra, then $X$ is a clonoid with source set $A$ and target algebra $B$ $\iff X$ is $(J_A, \text{Clo}(B))$-stable.
Basic Theory of Clonoids

- $X \subseteq \text{Fin}(A, B)$ is a clonoid from $A$ into $B = (B; G)$\n  $\iff$ there exists a two-sorted algebra $C = ((A, B); F \cup G)$ with $F \subseteq \text{Fin}(A, B)$ and $G \subseteq \text{Fin}(B, B)$ such that $X$ is the set of (finitary) term functions from $A$ to $B$.

- Let $R \subseteq A^l$, $S \subseteq B^l$, $X$ clonoid from $A$ to $(B; G)$.
  The pair $(R, S)$ is preserved by $X$
  $\iff$ $(R, S)$ is a subuniverse of $((A, B); X)^l \iff XR \subseteq S$.
  Then $(R, S)$ is an invariant relation of $X$. 

The “Polym-Inv”-Theorem

Let $A$ be finite, $f : A^n \to B$, $X$ a minion from $A$ to $B$. If $f$ preserves

$$(\{\pi_i^{(n)} \mid i \in \{1, \ldots, n\}\}, \ X^{[n]} \in P(A^A^n) \times P(B^A^n),)$$

then $f \in X$.

Hence each clonoid with finite source is determined by its finitary invariant relation pairs.
Basic Theory of clonoids

The “Inv-Polym”-Theorem [Pippenger 2002]

Let $A, B$ be finite, $R \subseteq A^k, S \subseteq B^k$. Let

$$X = \{ f : A^n \to B \mid n \in \mathbb{N}, \ f \ \text{preserves} \ (R, S) \}.$$

If $(C, D) \leq ((A, B), X)^m$, then $(C, D)$ can be obtained from $(R, S)$ using

- **direct products**

- **minors**: $\exists \sigma$ with

  $$\begin{align*}
  C &= \{(c_1, \ldots, c_m) \in A^m \mid (c_{\sigma(1)}, \ldots, c_{\sigma(k)}) \in R\} \\
  D &= \{(d_1, \ldots, d_m) \in B^m \mid (d_{\sigma(1)}, \ldots, d_{\sigma(k)}) \in S\}
  \end{align*}$$

- **projections**: $\exists \sigma$ with

  $$\begin{align*}
  C &= \{(c_{\sigma(1)}, \ldots, c_{\sigma(m)}) \mid (c_1, \ldots, c_k) \in R\} \\
  D &= \{(d_{\sigma(1)}, \ldots, d_{\sigma(m)}) \mid (d_1, \ldots, d_k) \in S\}
  \end{align*}$$

- **relaxations**: If $f : A^n \to B$ preserves $(R, S)$, then it also preserves $(R', S')$ with $R' \subseteq R, S \subseteq S'$. 
Occurrences of clonoids
Given: an algebra $A$.

Asked: describe the polynomial functions of $A$.

Example: On a finite field, every function is a polynomial function.
On the lattice $(\{0, 1\}, \lor, \land)$, every monotonic function is polynomial.
In describing $\text{Pol}(A) \subseteq \text{Fin}(A, A)$, one oftens describes **clonoids** inside $\text{Pol}(A)$.
Clonoids inside clones

Let $A$ be a universal algebra, and let $\rho \in \text{Con}(A)$, $o \in A$.

$P_0$ = those functions that fix $o = \text{Polym}({o}, \{o\})$.

All functions that map into $o/\rho = \text{Polym}(A, o/\rho)$. This is a $(\text{Pol}(A), P_0)$-stable class.

For submodules $I, J$ of an $R$-module $M$, the Noetherian quotient $(I : J) = \{r \in R \mid rJ \subseteq I\}$ “is” $\text{Polym}(J, I) \cap \text{Clo}^{[1]}(R M)$.

All functions that are constant on $\rho$-cosets = $\text{Polym}(\rho, =_A)$. This is a $(\text{Pol}(A), \text{Pol}(A))$-stable class.

All functions that map into one $\rho$-class = $\text{Polym}(A \times A, \rho)$. This is a $(\text{Pol}(A), \text{Pol}(A))$-stable class.

For example, [EA, 2006] describes the polynomial functions of the finite nonsolvable special linear groups.
Occurrences in clone theory

We let $A, B$ be abelian groups, $f : A^n \to B$.

- For $a \in A^n$, $\Delta_a(f)(x) := f(x + a) - f(x)$.
- $F\text{DEG}(f) :=$ the minimal $k \in \mathbb{N}_0$ with $\Delta_{a_1}\Delta_{a_2} \cdots \Delta_{a_{k+1}} f = 0$ for all $a_1, \ldots, a_{k+1} \in A^n$.

- **Intuitive:** $f : \mathbb{R}^1 \to \mathbb{R}$ is a polynomial of degree $\leq 2 \iff f''' = 0$.

By [Leibman 2002], we have

**Lemma**

Let $\text{End}(A)$ be the set of all endomorphisms from $A^n$ to $A$, $n \in \mathbb{N}$.

Then for $k \in \mathbb{N}_0$,

$$D_k := \{ f : A^n \to B \mid \text{FDEG}(f) \leq k \}$$

is an $(\text{End}(A), \text{End}(B))$-stable class.
Clonoids in universal algebraic geometry

Let $S \subseteq \bigcup_{n\in \mathbb{N}} P(A^n)$ be a system of finitary relations on $A$. Then:

- If for all $n, k, \sigma : [n] \to [k]$, $B \in S$ with $B \subseteq A^n$, 
  \[
  B_\sigma := \{(s_1, \ldots, s_k) \in A^k \mid (s_{\sigma(1)}, \ldots, s_{\sigma(n)}) \in B\} = \{s \in A^k \mid s \circ \sigma \in B\}
  \]
satisfies $B_\sigma \in S$, then the set of characteristic functions
  \[
  C_S := \{\chi_B \mid B \in S\}
  \]
is a clonoid from $A$ into $\{0, 1\}$ called the **characteristic clonoid** of $S$. $B_\sigma$ is a **minor** of $B$.

- If $S$ is closed under intersections, $C_S$ is a clonoid from $A$ into $(\{0, 1\}, \land)$.
- If $S$ is closed under intersections and unions, $C_S$ is a clonoid from $A$ into $(\{0, 1\}, \land, \lor)$. 


We look for an algorithm that does the following:

**Input:** A finite graph $G$.

**Output:** Yes if $G$ is 3-colorable. No if $G$ is not even 100-colorable.

Do not bother about borderline cases ($G$ is 100-colorable and not 3-colorable).

This is $\text{PCSP}(K_3, K_{100})$, where $K_n = ([n], \neq)$.

**Theorem [Pippenger 2002; Brakensiek, Guruswami 2019; Barto, Bulín, Krokhin, Opršal 2019]**

If $\text{Polym}(A, B) \subseteq \text{Polym}(A', B')$, then $\text{PCSP}(A', B')$ is polynomial-time reducible to $\text{PCSP}(A, B)$. 
The equational theory of $W$ in $A$ [EA and Mayr 2016]

$A$ algebra, $W$ subvariety of $\mathbb{V}(A)$.

$$\text{Th}_A(W) := \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^A(a) \\ t^A(a) \end{pmatrix} \mid k \in \mathbb{N},$$

$s, t$ are $k$-variable terms in the language of $A$ with $W \models s \approx t$).

$\text{Th}_A(W)$ is a clonoid with source set $A$ and target algebra $A \times A$.

These clonoids from $A$ into the algebra $A \times A$ can distinguish all subvarieties of $\mathbb{V}(A)$. 
The finite relatedness result
Finite relatedness of Mal’cev clonoids

Theorem [EA Mayr McKenzie 2014, 2016]
Let $A$ be finite, $B$ be a finite algebra with a Mal’cev term, and let $C$ be a clonoid from $A$ to $B$. Then

1. There are $n \in \mathbb{N}$, $R \subseteq A^n$, $S \subseteq B^n$ such that $C = \text{Polym}(R, S)$.
2. There is no infinite descending chain of clonoids from $A$ to $B$.

Corollary
Every subvariety of a finitely generated variety with Mal’cev term is finitely generated.

We will now explain the main steps in the proof.
Definition of Forks
Let $A$ be an algebra, let $m \in \mathbb{N}$, and let $F$ be a subuniverse of $A^m$. For $i \in \{1, \ldots, m\}$, we define the relation $\text{Forks}_i(F)$ on $A$ by

$$\text{Forks}_i(F) := \{ (a_i, b_i) \mid (a_1, \ldots, a_m) \in F, (b_1, \ldots, b_m) \in F, (a_1, \ldots, a_{i-1}) = (b_1, \ldots, b_{i-1}) \}.$$ 

If $(c, d) \in \text{Forks}_i(F)$, we call $(c, d)$ a fork of $F$ at $i$.

If

$$u = (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m) \in F \quad \text{and} \quad v = (a_1, \ldots, a_{i-1}, d, b_{i+1}, \ldots, b_m) \in F,$$

then $(u, v)$ is a witness of the fork $(c, d)$ at $i$. 

Forks
Forks in linear algebra

Let $F \leq (\mathbb{Z}_5, +)^4$ with $F = \langle (1, 2, 2, 2), (3, 1, 4, 2), (1, 2, 0, 3) \rangle$. When we compute the row echelon form of

$$
\begin{pmatrix}
1 & 2 & 2 & 2 \\
3 & 1 & 4 & 2 \\
1 & 2 & 0 & 3
\end{pmatrix}
\xrightarrow{}
\begin{pmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

we obtain that $(0, 1)$ is a fork of $F$ at 1 with witness $(0, 0, 0, 0), (1, 2, 0, 3)$ and that $(0, 1)$ is a fork of $F$ at 3 with witness $(0, 0, 0, 0), (0, 0, 1, 2)$. 
Significance of forks

Let $A$ be an algebra with a Mal’cev term, and let $F \leq A^m$.

- Let $G \subseteq F$ be such that $G$ contains at least one witness for each fork of $F$. Then $G$ generates $F$.

Hence for $a := |A| < \aleph_0$, $F$ can be generated by $a^2 \cdot m$ elements and thus

$$\# \text{Sub}(A^m) \leq \left( \frac{a^m}{a^2 m} \right) \leq a^m \cdot a^2 \cdot m \leq 2^{O(a)m^2}.$$

(The Fork Lemma)

If $E \leq F$ and every fork of $F$ is a fork of $E$, then $E = F$. 
We represent a clonoid \( C \) with source set \( A = \{a_1, \ldots, a_t\} \) and target algebra \( B \) using forks.

For \( a \in A^n \), we define the forks of \( C \) at \( a \) by

\[
\text{Forks}(C, a) := \{(f_1(a), f_2(a)) \in B \times B \mid f_1 \in C, f_2 \in C, f_1(z) = f_2(z) \text{ for all } z \in A^n \text{ with } z <_{\text{lex}} a\}.
\]

Fork Lemma for Clonoids

A finite set, \( B \) finite algebra with Mal’cev term, \( C, D \) clonoids from \( A \) to \( B \). If \( C \subseteq D \) and \( C \) and \( D \) have the same forks, then \( C = D \).
Connections between forks of different arity

\[ a \leq_E b :\iff b \text{ can be obtained from } a \text{ by inserting additional letters anywhere after their first occurrence in } a. \] Such orders are studied further in [McDevitt 2018].

\[ \text{abac} \leq_E \text{aaababbaabaccba} \]

**Embedded Forks Lemma ([EA Mayr McKenzie 2014] for clones)**

Let \( A \) be a set and \( B \) be an algebra, let \( C \) be a clonoid from \( A \) to \( B \), and let \( a, b \in A^* \) with \( a \leq_E b \).

Then \( \text{Forks}(C, b) \subseteq \text{Forks}(C, a) \).
Facts on the embedding orderings

For a finite set $A$, $(A^*, \leq_E)$ has no infinite descending chains and no infinite antichains. In other words, $(A^*, \leq_E)$ is well partially ordered. Let $U$ be the set of upward closed subsets of $(A^*, \leq_E)$. Then

- $(U, \subseteq)$ has no infinite ascending chains.

  **Reason:** $\bigcup_{i \in \mathbb{N}} U_i$ is generated by its minimal elements $M \subseteq A^*$. As an antichain in $(A^*, \leq_E)$, $M$ is finite.

  Hence there is $j \in \mathbb{N}$ with $M \subseteq U_j$, and thus $U_{j+1} \subseteq U_j$.

- $(U, \subseteq)$ has no infinite antichains. Well-known; a proof in [EA and Aichinger, Expo. Math. (2020)].
Proof of the finite relatedness result

Theorem [EA Mayr McKenzie 2014, 2016]
Let $A$ be finite, $B$ be a finite algebra with a Mal’cev term. Then there is no infinite descending chain of clonoids from $A$ to $B$.

Proof (sketch):
▶ Let $(C_i)_{i \in \mathbb{N}}$ be an infinite descending chain of clonoids from $A$ to $B$.
▶ For each $D \leq B \times B$, $U_D(C_i) := \{ a \in A^* \mid \text{Forks}(C_i, a) \subseteq D \}$ is upward closed by the Embedded Forks Lemma.
▶ There is $D \subseteq B \times B$ such that $(U_D(C_i))_{i \in \mathbb{N}}$ is an infinite ascending chain. Contradiction.
A “constructive” version

Theorem [EA Mayr McKenzie 2014, 2016]
Let $A$ be finite, $B$ be a finite algebra with a Mal’cev term, and let $C$ be a clonoid from $A$ to $B$. Let

$$m := \max\{|a| \mid \exists D \leq B \times B : a \text{ is minimal in } U_D(C)\}.$$ 

Then $C = \text{Polym}(R; S)$ with $R \subseteq A^{\lvert A \rvert^m}$ and $S \subseteq B^{\lvert A \rvert^m}$.

We do not know how to compute $m$. Computing $m$ would allow us to decide:

**Given:** $F \subseteq_{\text{fin}} \text{Fin}(B, B)$, $k \in \mathbb{N}$, $\rho \subseteq B^k$.

**Asked:** $\text{Clo}_B(F \cup \{d\}) = \text{Polym}(\rho)$.

Note that $\subseteq$ is easy to check.
Open problems - clones and clonoids

1. Release finiteness of $A$ or $B$.
2. $A$ finite, $B$ Mal’cev algebra. Is there an infinite antichain of clonoids from $A$ to $B$?
3. Is there an infinite antichain of clones with a Mal’cev term on a finite set?
Significance of “no antichains”

We imitate [Robertson and Seymour, *Graph minors. XX. 2004*].

**Fact**

A finite, \(d\) Mal’cev, \(f, g\) operations on \(A\).

**Suppose that there is no infinite antichain of clones on \(A\) containing \(d\).** We define

\[
f \leq_d g :\iff f \in \text{Clo}_A(g, d).
\]

Let \(\psi\) be a property of operations such that

\[
g \models \psi, f \leq_d g \Rightarrow f \models \psi.
\]

Then \(\psi\) can be decided in polynomial time in \(||f|| \sim |A|^{\text{arity}(f)}\).

**Proof:** \(\psi\) has finitely many minimal counterexamples \(g_1, \ldots, g_k\). The property \(g_i \in \text{Clo}_A(f, d)\) can be checked “easily”.
The “no descending chains” result for clonoids yields:

**Theorem [EA and Mayr 2016]**

Let $A$ be a finite algebra with a Mal'cev term. Then every subvariety of $\mathbb{V}(A)$ is generated by a finite algebra.

**Open problems:**

- Given $A$, $B$ similar finite algebras with Mal'cev term, $\mathbb{V}(A) \cap \mathbb{V}(B)$ is therefore finitely generated. Can you give an upper bound for the size of a generator?
- Is there a finitely generated variety with a Mal'cev term (of finite type) with an infinite antichain of subvarieties? (Infinite ascending chains cannot exist, and infinite descending chains sometimes do exist; pointed group [Bryant 1982]).
Description of concrete clonoids
Clonoids closed under near-unanimity terms

\[ t(x_1, \ldots, x_n) \text{ is NU-term} \iff t(x, y, y, \ldots, y) = t(y, x, y, \ldots, y) = \cdots = t(y, y, \ldots, x) = y \text{ for all } x, y. \]

**Theorem [Baker and Pixley 1975; Sparks 2019]**

A finite, \( \mathbf{B} \) algebra with \( n \)-ary NU-term Let \( C, D \) clonoids from \( A \) to \( \mathbf{B} \) that have the same functions of arity \( |A|^{n-1} \). Then \( C = D \).

Remark: For clones, \( |A|^{n-1} \) can be improved [Lakser, 1989], [Kerkhoff 2011], [Kerkhoff and Zhuk, 2014].
Clonoids closed under near-unanimity terms: a logical consequence

For a set of f.o. formulas \( \Phi \) in the language of the f.o. structure \( A \), let

\[
\text{Def}^{[n]}(\Phi) := \left\{ \{(a_1, \ldots, a_n) \in A^n \mid A \models \varphi(a_1, \ldots, a_n)\} \mid \varphi \in \Phi \text{ with } \text{freeVars}(\varphi) \subseteq \{x_1, \ldots, x_n\} \right\}.
\]

Corollary [EA and Rossi 2020]

Let \( A_1 \) and \( A_2 \) be f.o. structures on a finite set \( A \). For each \( i \in \{1, 2\} \), let \( \Phi_i \) be a set of f.o. formulas in the language of \( A_i \) that is closed under \( \land, \lor \), and substituting variables. Then \( \text{Def}(A_1, \Phi_1) = \text{Def}(A_2, \Phi_2) \) if and only if \( \text{Def}^{[|A|^2]}(A_1, \Phi_1) = \text{Def}^{[|A|^2]}(A_2, \Phi_2) \).
By Sparks’ Theorem, there are only finitely many clonoids from $A$ into an algebra with NU-term.

**Open problem [Sparks 2019]**

Is there a finite set $A$ and an algebra $B$ such that

- there are only finitely many clonoids from $A$ to $B$ and
- $|A| > 1$ and $B$ has no NU-term?
Clonoids with 2-element target

Theorem [Sparks 2019]
For $|B| = 2$ and finite $A$, let $c$ be the number of clonoids from $A$ to $B$. Then:

1. $c < \aleph_0 \iff B$ has an NU-term.
2. $c = \aleph_0 \iff B$ has a Mal’cev term and no NU-term.
3. $c = 2^{\aleph_0} \iff B$ has neither NU nor Mal’cev term.
Linearly closed clonoids

Motivation: Bulatov and Idziak described all extensions of

$$\text{Pol}(\mathbb{Z}_p \times \mathbb{Z}_p, +) = \text{Clo}(\mathbb{Z}_p \times \mathbb{Z}_p, +, 1).$$

We try to describe extensions of $\text{Clo}(\mathbb{Z}_p \times \mathbb{Z}_p, +)$. 
Linear closed clonoids from $\mathbb{Z}_p$ to $\mathbb{Z}_p$

First step:

**Theorem [Kreinecker 2020]**

Let $L := \text{Clo}(\mathbb{Z}_p, +)$. For every $(L, L)$-stable subclass $C$ of $\text{Fin}(\mathbb{Z}_p, \mathbb{Z}_p)$, there is $M \subseteq \mathbb{N}_0$ such that

- $\forall n \in \mathbb{N} : (n \in M \land n > p - 1) \Rightarrow n - (p - 1) \in M$.
- $f \in C \iff f$ is induced by a polynomial such that the total degree of each monomial is in $M$.

The correspondence $C \mapsto M$ is a lattice isomorphism from the set $(L, L)$-stable classes to the subsets of $\mathbb{N}_0$ satisfying the first condition.

**Example:** $p = 3$, $M = \{6, 4, 2\} \cup \{11, 9, 7, 5, 3, 1\}$ is minor closed, $x_1x_2x_8x_9x_{10}^2 + x_1^2x_2^2 - x_1x_2x_3x_4x_5x_6x_7 \in C$.

Hence the lattice of $(L, L)$-stable classes is isomorphic to $(\mathbb{N}_0 \cup \{\infty\}, \subseteq)^{p-1} \times 2$-element chain.
Embedding the clonoid into a clone

Theorem [Fioravanti and Kreinecker 2020]
For an \((L, L)\)-stable subclass \(C\) of \(\text{Fin}(\mathbb{Z}_p, \mathbb{Z}_p)\), define

\[
\phi(C) := \{((x_1, y_1), \ldots, (x_n, y_n)) \mapsto (\sum_{i=1}^{n} a_ix_i + c(y_1, \ldots, y_n), \sum_{i=1}^{n} b_iy_i) \mid a_i, b_i \in \mathbb{Z}, c \in C\} \subseteq \text{Fin}(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p).
\]

Then \(\phi(C)\) is a clone. The mapping \(C \mapsto \phi(C)\) is a lattice embedding.

Theorem [Kreinecker 2020]
Let \(p > 2\) be a prime. Then there are infinitely many not finitely generated clones on \(\mathbb{Z}_p \times \mathbb{Z}_p\) which contain \(+\).
Extensions of $\text{Clo}(\mathbb{Z}_p \times \mathbb{Z}_q, +)$

Theorem [Fioravanti 2020]
Let $s \in \mathbb{N}$ be squarefree. Then each clone containing $\text{Clo}(\mathbb{Z}_s, +)$ is generated by its functions of arity $\leq s$.

Lemma [Fioravanti 2019]
Let $L_p := \text{Clo}(\mathbb{Z}_p, +)$ and $L_q := \text{Clo}(\mathbb{Z}_q, +)$. Then every $(L_p, L_q)$-stable subclass of $\text{Fin}(\mathbb{Z}_p, \mathbb{Z}_q)$ is generated by one unary function.
¡Muchas gracias por su atención y esta invitación muy apreciada!