Punctured and Structured Nullstellensätze

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Nullstellensätze

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Theorem (Hilbert 1893).

Let $f_1, \ldots, f_s, g \in \mathbb{C}[x_1, \ldots, x_n]$. Then g vanishes on all common zeros of $f_1, \ldots, f_n$ iff there are $a_1, \ldots, a_s \in \mathbb{C}[\boldsymbol{x}]$ and $r \in \mathbb{N}$ such that $g^r = a_1 f_1 + \cdots a_s f_s$.

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Theorem (Clark's Finitesatz, 2014).

Let $\mathbb{F}$ be a field, let $f_1, \ldots, f_r, g \in \mathbb{F}[x_1, \ldots, x_n]$, and let $X \subseteq_{\text{fin}} \mathbb{F}^n$. Then g vanishes on all common zeros of $f_1, \ldots, f_n$ in X iff there are $a_1, \ldots, a_s, h \in \mathbb{F}[\boldsymbol{x}]$ such that

$$g = a_1 f_1 + \cdots a_r f_r + h$$

and h vanishes on X.

Combinatorial Nullstellensätze

Alon's Combinatorial Nullstellensatz I

Theorem (Alon's Nullstellensatz I).

Let $\mathbb{K}$ be a field, $S = \bigtimes_{i=1}^{n} S_i$ with $S_i \subseteq_{\text{fin}} \mathbb{K}$. Then $f \in \mathbb{K}[\boldsymbol{x}]$ vanishes on S iff there are $a_1, \ldots, a_s \in \mathbb{K}[\boldsymbol{x}]$ such that

$$f = a_1 g_1 + \cdots + a_r g_r,$$

where $g_i = \prod_{a \in S_i} (x_i - a)$ and $\deg(a_i g_i) \leq \deg(f)$ for all i.

Alon's Combinatorial Nullstellensatz II

Theorem (Alon's Combinatorial Nullstellensatz II).

Let $\mathbb{K}$ be a field, $S = \bigtimes_{i=1}^{n} S_i$ with $S_i \subseteq_{\text{fin}} \mathbb{K}$.
Let $f \in \mathbb{K}[\boldsymbol{x}]$ be such that f contains a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i < |S_i|$ for all i.
If for all monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ of f with $\alpha \neq \gamma$ we have

$$\sum_{i=1}^{n} \gamma_i \leq \sum_{i=1}^{n} \alpha_i, \qquad \text{(Alon's Condition)}$$

then there is $\boldsymbol{s} \in S$ with $f(\boldsymbol{s}) \neq 0$.

Improvements: Replace (Alon's Condition) with weaker conditions.

Improved Combinatorial Nullstellensatz II

Theorem (Combinatorial Nullstellensatz II).

Suppose that $f \in \mathbb{K}[\boldsymbol{x}]$ contains a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i < |S_i|$ for all i. If for all monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ of f with $\alpha \neq \gamma$ we have

$$\sum_{i=1}^{n} \gamma_i \leq \sum_{i=1}^{n} \alpha_i, \qquad \text{(Alon's Condition)}$$

then there is $\boldsymbol{s} \in S$ with $f(\boldsymbol{s}) \neq 0$.

Improvements: Replace (Alon's Condition) with the following weaker conditions.

1. (Tao-Vu-Lasoń's Condition 2006) $\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1]$.
2. (Schauz's Condition 2008) $\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, |S_i| - 1]$.

Structured Grids

Structured Grids

Definition (Nica 2023).
$S \subseteq_{\text{fin}} \mathbb{K}$ is λ-null $:\Leftrightarrow$ in $\prod_{a \in S}(x-a)$, the coefficients of $x^{|S|-1}, \ldots, x^{|S|-\lambda}$ are zero.
Examples

- Every finite S is 0-null.
- $\{x \in \mathbb{C} \mid x^n = 1\}$ is $n-1$-null.
- $\{0\}$, $\varnothing$ are μ-null for all $\mu \in \mathbb{N}$.
- S is 1-null if $\sum_{a \in S} a = 0$.

Theorem (Nica 2023).

Let $\mathbb{K}$ be a field, $S = \bigtimes_{i=1}^{n} S_i$ such that $S_i \subseteq_{\text{fin}} \mathbb{K}$ and S_i **is λ_i-null**.
Let $f \in \mathbb{K}[\boldsymbol{x}]$ be such that f contains a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i < |S_i|$ for all i.
If for all monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ of f with $\alpha \neq \gamma$ we have

$$\sum_{i=1}^{n} \gamma_i \leq \min(\lambda_1, \ldots, \lambda_n) + \sum_{i=1}^{n} \alpha_i, \qquad \text{(Nica's Condition)}$$

then there is $\boldsymbol{s} \in S$ with $f(\boldsymbol{s}) \neq 0$.

Improvements:

- (EA-Schmitt-Zhan's Condition)
  $\exists i \in \underline{n} : \gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, \max(|S_i| - 1, \alpha_i + \lambda_i)]$.

Comparison of the Nullstellensätze

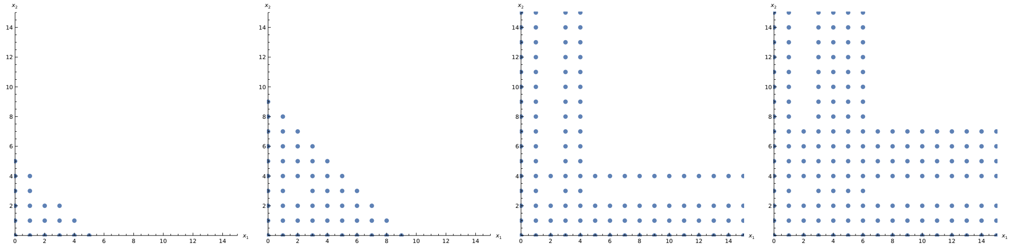
These theorems have in common:

- they guarantee a nonzero in a grid.
- the condition ensuring this is:
  1. there is a monomial $\boldsymbol{x}^\alpha$ in f with $\alpha_i < |S_i|$ for all i.
  2. all other monomials $\boldsymbol{x}^\gamma$ of f are innocuous.

The more monomials one can declare innocuous, the better.

Comparison of the Nullstellensätze

Example. $S = \{(a, b) \in \mathbb{C}^2 \mid a^5 = b^5 = 1\}$, $\lambda_1 = \lambda_2 = 4$. Suppose f contains the monomial $x_1^2 x_2^3$. Then the following monomials are declared innocuous:



Alon | Nica | Schauz | EA-Schmitt-Zhan

Figure: $x_1^2 x_2^3$ + any linear combination of the dotted monomials does not vanish on $S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1^5 = x_2^5 = 1\}$.

Improved Nullstellensätze

Generalisations and Improvements:

- Multiplicity: $\boldsymbol{c}$ is a t-fold zero of f if all monomials of $f' := f(c_1 + x_1, \ldots, c_n + x_n)$ have total degree at least t. Ball and Serra (2009) provide theorems with bottom line:
  "Then there is $\boldsymbol{s} \in S$ such that $\boldsymbol{s}$ is not a t-fold zero of f."
- Multisets (Kós and Rónyai 2012).
- Beyond grids: Punctured Grids $X \setminus Y$, where X, Y are grids. (Ball and Serra 2009)
- Structured grids: Use the property that an edge of the grid is λ-null. (Nica 2023)

Our recent manuscript provides combinations of these, for example a

Structured Nullstellensatz for **punctured** grids.

Manuscript: E.Aichinger, J.R.Schmitt, H.Zhan, *Structured and punctured Nullstellensätze*, arxiv 2025.

Structured Nullstellensätze for punctured grids

Theorem (A structured Nullstellensatz for punctured grids), EA-Schmitt-Zhan 2025.

Let $X = \bigtimes_{i=1}^{n} X_i, Y = \bigtimes_{i=1}^{n} Y_i$ be grids over $\mathbb{K}$ with $Y_i \subseteq X_i$ and X_i, Y_i λ-null for all i. Let

$$P := X \setminus Y.$$

Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ with a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that

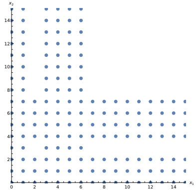
1. for all i : $\alpha_i < |X_i|$,
2. there exists i such that $\alpha_i < |X_i| - |Y_i|$,
3. $\sum_{i=1}^{n} \alpha_i \geq \deg(f) - \lambda$.

Then there is $\boldsymbol{z} \in P$ with $f(\boldsymbol{z}) \neq 0$.

Proofs

Proof Ideas

- Given $S \subseteq_{\text{fin}} \mathbb{K}^n$, find generators G of the ideal $\mathbb{I}(S) = \{f \in \mathbb{K}[\boldsymbol{x}] \mid f(\boldsymbol{a}) = 0 \text{ for all } \boldsymbol{a} \in S\}$.
- We want to show that $f \notin \mathbb{I}(S)$.
- Show that $\boldsymbol{x}^\alpha$ cannot disappear during multivariate polynomial division by G because
  - $\boldsymbol{x}^\alpha$ cannot be reduced by G.
  - All other monomials $\boldsymbol{x}^\gamma$ cannot produce $\boldsymbol{x}^\alpha$ in the course of the division – not in the first and not in any further step.



- f has nonzero remainder by G: Then $f \notin \langle G \rangle$ if G is a Gröbner basis.
- Use the S-Polynomial Theorem (Buchberger 1965) to show that G is indeed a Gröbner basis.

Lower bounds for the number of nonzeros

Alon-Füredi Nonzero Counting Theorem for punctured grids

Theorem (Alon-Füredi for punctured grids), EA-Schmitt-Zhan 2025.

Let $X = \bigtimes_{i=1}^n X_i$ and $Y = \bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i \subseteq X_i$ for all i, $P := X \setminus Y$, $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \{0\}$.
Let $a_i := |X_i|$, $b_i := |Y_i|$ and

$$A := \{(y_1, \ldots, y_n) \in \mathbb{N}^n \mid \forall i \in \underline{n} : 1 \leq y_i \leq a_i,\ \exists i \in \underline{n} : y_i > b_i, \text{ and } \sum_{i=1}^n y_i \geq \sum_{i=1}^n a_i - \deg(f)\}.$$

If $P \setminus \mathbb{V}(f) \neq \varnothing$, then

$$|P \setminus V(f)| \geq \min\{\prod_{i=1}^n y_i - \prod_{i=1}^n \min(y_i, b_i) \mid (y_1, \ldots, y_n) \in A\}.$$

Alon-Füredi Nonzero Counting Theorem for punctured grids

The proof is based on:

Clark's Monomial Alon-Füredi Theorem (Clark 2024).

Let X be a finite subset of $\mathbb{K}^n$, let $f \in \mathbb{K}[x_1, \ldots, x_n]$, and let $g \in \mathbb{I}(X) + \langle f \rangle$ with $g \neq 0$. Then

$$|X \setminus \mathbb{V}(f)| \geq |\,\Delta(\mathbb{I}(X)) \cap \{\textsc{Lm}(g)\}{\uparrow}\,|.$$

For $G \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and an admissible monomial ordering $\leq_a$, we define

$$\begin{aligned} G{\uparrow} \quad &:= \quad \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n \text{ and } \exists g \in G : \textsc{Lm}(g) \text{ divides } \boldsymbol{x}^\alpha\}, \\ \Delta(G) \quad &:= \quad \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n\} \setminus (G{\uparrow}) = \\ &\quad\;\; \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n \text{ and there is no } g \in G \text{ such that } \textsc{Lm}(g) \text{ divides } \boldsymbol{x}^\alpha\}. \end{aligned}$$