Subvarieties and Clones

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Main question in this talk

Question

Are subvarieties of finitely generated varieties again finitely generated?

Answer

Sometimes.

Goal

Improve.

Classes of algebras

We will study:

- *classes of algebras* of with the same operation symbols (of the same *type*) $\mathcal{F}$.
- Example: $\mathcal{F} := \{\cdot, {}^{-1}, 1\}$, $K :=$ class of all groups.
- *identities*: $s(x_1, \ldots, x_k) \approx t(x_1, \ldots, x_k)$.
- Example:
  $\Phi = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ 1 \cdot x \approx x,\ x^{-1} \cdot x \approx 1,\ x^6 \approx y^{15}\}$.
- *Validity of identities* in an algebra **A** of type $\mathcal{F}$.
- Example: $\mathbf{A} \models \Phi \Leftrightarrow \mathbf{A}$ is a group of exponent 1 or 3.

Varieties

Theorem [Birkhoff, 1935, Theorem 10]

Let K be a nonempty class of algebras of the same type $\mathcal{F}$. TFAE:

1. $\exists$ set of identities $\Phi : K = \{\mathbf{A} \,|\, \mathbf{A}$ is of type $\mathcal{F}$ and $\mathbf{A} \models \Phi\}$.
   (*Meaning: K* can defined using identities.)
2. K is closed under taking
   - $\mathbb{H}$ homomorphic images
   - $\mathbb{S}$ subalgebras
   - $\mathbb{P}$ cartesian products.

A class K of algebras that can be defined by a set of identities is called a *variety*.

Finitely generated varieties

Definition

$\mathbf{A}$ algebra. $\mathbb{V}(\mathbf{A}) :=$ the smallest variety that contains $\mathbf{A}$.

Theorem

$\mathbb{V}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A})$.

Theorem

$\mathbf{B} \in \mathbb{V}(\mathbf{A})$ if and only if $\forall s, t : \mathbf{A} \models s \approx t \;\Rightarrow\; \mathbf{B} \models s \approx t$.

Definition

A variety V is *finitely generated* $:\Leftrightarrow$ there is a finite algebra $\mathbf{A}$ with $V = \mathbb{V}(\mathbf{A})$.

Finite Generation of Subvarieties

Theorem (follows from [Jónsson, 1967])

Let $\mathbf{L}$ be a finite lattice. Then every subvariety of $\mathbb{V}(\mathbf{L})$ is finitely generated.

Proof: $\mathbb{V}(\mathbf{L})$ contains, up to isomorphism, only finitely many subdirectly irreducible lattices (Jónsson's Lemma).

Theorem [Oates and Powell, 1964]

Let $\mathbf{G}$ be a finite group. Then every subvariety of $\mathbb{V}(\mathbf{G})$ is finitely generated.

Proof: $\mathbb{V}(\mathbf{G})$ contains, up to isomorphism, only finitely many groups $\mathbf{H}$ with $\mathbf{H} \notin \mathbb{V}(\{\mathbf{A} \mid \mathbf{A} \in \mathbb{V}(\mathbf{H}), |\mathbf{A}| < |\mathbf{H}|\})$. (Long proof using "critical groups".)

Note that both $\mathbb{V}(\mathbf{G})$ and $\mathbb{V}(\mathbf{L})$ contain only *finitely many* subvarieties.

Finite Generation of Subvarieties

Theorem [Bryant, 1982]

There is an expansion of a finite group with one constant operation such that the variety generated by this algebra has infinitely many subvarieties.

They might all be finitely generated, though.

Theorem [Oates MacDonald and Vaughan-Lee, 1978]

There is a three-element algebra $\mathbf{M} = (M, *, c)$ such that $\mathbb{V}(\mathbf{M})$ has subvarieties that are not f.g.

Recognizing f.g. subvarieties

Lemma [Oates MacDonald and Vaughan-Lee, 1978]

V f.g. variety. TFAE:

1. The subvarieties of V, ordered by $\subseteq$, satisfy (ACC).
2. Every subvariety of V is f.g.

Lemma

V f.g. variety. TFAE:

1. The subvarieties of V, ordered by $\subseteq$, satisfy (DCC).
2. For every subvariety W of V there is a finite set of identities Φ with $W = \{\mathbf{A} \in V \mid \mathbf{A} \models \Phi\}$. ($W$ is finitely based relative to V.)

Equational theory of W in $\mathbf{A}$

Definition [Aichinger and Mayr, 2014]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathbf{Th}_{\mathbf{A}}(W) := \{(a_1,\dots,a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

s, t are k-variable terms in the language of $\mathbf{A}$

with $W \models s \approx t\}$.

Examples

1. $\mathbf{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})) = \{(t,t) \mid t \in \mathrm{Clo}(\mathbf{A})\}$.
2. $\mathbf{A} := \mathbf{S}_3$, $W := \{\mathbf{G} \in \mathbb{V}(\mathbf{S}_3) \mid \mathbf{G}$ is abelian$\}$. Then $((\begin{smallmatrix}\pi_1 \\ \pi_2\end{smallmatrix}) \mapsto \begin{pmatrix} \pi_1^{-1} \circ \pi_2 \circ \pi_1 \\ \pi_2 \end{pmatrix}) \in \mathbf{Th}_{\mathbf{S}_3}(W)$.
3. $W :=$ class of one element algebras of type $\mathcal{F}$. Then $\mathbf{Th}_{\mathbf{A}}(W) = \{(s,t) \mid k \in \mathbb{N},\ s,t \in \mathrm{Clo}_k(\mathbf{A})\}$.

Distinguishing subvarieties of $\mathbb{V}(\mathbf{A})$ inside A

Lemma

$\mathbf{A}$ be algebra, W_1 and W_2 subvarieties of $\mathbb{V}(\mathbf{A})$. Then we have:

$$W_1 \subseteq W_2 \text{ if and only if } \mathbf{Th}_{\mathbf{A}}(W_2) \subseteq \mathbf{Th}_{\mathbf{A}}(W_1).$$

Clonoids

What is

$$\mathbf{Th_A}(W) = \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

s, t are k-variable terms in the language of $\mathbf{A}$

with $W \models s \approx t\}$?

$\mathbf{Th_A}(W)$ is a *clonoid with source set A and target algebra $\mathbf{A} \times \mathbf{A}$.*

Definition of Clonoids

$$
\begin{array}{lll}
A & \ldots & \text{set} \\
\mathbf{B} & \ldots & \text{algebra} \\
\mathbf{C} & \ldots & \text{finitary functions from } A \text{ to } B \\
\mathbf{C} & \subseteq & \bigcup_{n\in\mathbb{N}} B^{A^n} \\
\mathbf{C}^{[k]} & := & \mathbf{C} \cap B^{A^k} (k\text{-ary functions in } \mathbf{C}).
\end{array}
$$

Definition of Clonoids

Definition

$\mathbf{B}$ algebra, A nonempty set, $\mathbf{C} \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$. $\mathbf{C}$ is a *clonoid with source set A and target algebra* $\mathbf{B}$ if

1. for all $k \in \mathbb{N}$: $\mathbf{C}^{[k]}$ is a subalgebra of $\mathbf{B}^{A^k}$, and
2. for all $k, n \in \mathbb{N}$, for all $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$, and for all $c \in \mathbf{C}^{[k]}$, the function $c' : A^n \to B$ defined by

$$c'(a_1, \ldots, a_n) := c(a_{i_1}, \ldots, a_{i_k})$$

satisfies $c' \in \mathbf{C}^{[n]}$.

Representation of Clonoids

We represent a clonoid **C** with source set $A = \{a_1, \dots, a_t\}$ and target algebra **B** using forks.

Definition (forks of $\mathbf{B}^{A^n}$ at $\mathbf{a}$)

For $\mathbf{a} \in A^n$, let

$$\varphi(\mathbf{C}, \mathbf{a}) := \{ (f_1(\mathbf{a}), f_2(\mathbf{a})) \in B \times B \mid f_1(\mathbf{z}) = f_2(\mathbf{z}) \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\mathrm{lex}} \mathbf{a} \}.$$

Use of forks

- [Berman et al., 2010, Corollary 3.9]
- [Aichinger, 2000, Proof of Proposition 3.1]

Mal'cev and edge terms

Mal'cev terms

A ternary term t is a *Mal'cev term* on $\mathbf{A}$ if for all $a, b \in A$:

$$t^{\mathbf{A}}(a, a, b) = t^{\mathbf{A}}(b, a, a) = b.$$

Edge terms

For $k \geq 3$, a $(k+1)$-ary term is a *k-edge term* on $\mathbf{A}$ if for all $a, b \in A$:

$$\begin{aligned}
t^{\mathbf{A}}(a, a, b, b, b, \ldots, b) &= b \\
t^{\mathbf{A}}(b, a, a, b, b, \ldots, b) &= b \\
t^{\mathbf{A}}(b, b, b, a, b, \ldots, b) &= b \\
&\ddots \\
t^{\mathbf{A}}(b, b, b, b, b, \ldots, a) &= b
\end{aligned}$$

(still wrong!)

Mal'cev and edge terms

Mal'cev terms

A ternary term t is a *Mal'cev term* on $\mathbf{A}$ if for all $a, b \in A$:

$$t^{\mathbf{A}}(a,a,b) = t^{\mathbf{A}}(b,a,a) = b.$$

Edge terms

For $k \geq 3$, a $(k+1)$-ary term is a k*-edge term* on $\mathbf{A}$ if for all $a, b \in A$:

$$\begin{aligned}
t^{\mathbf{A}}(a,a,b,b,b,\dots,b) &= b\\
t^{\mathbf{A}}(a,b,a,b,b,\dots,b) &= b\\
t^{\mathbf{A}}(b,b,b,a,b,\dots,b) &= b\\
&\ddots\\
t^{\mathbf{A}}(b,b,b,b,b,\dots,a) &= b
\end{aligned}$$

Edge terms and few subpowers

Theorem [Berman et al., 2010]

A finite algebra. **A** has an edge term $\Leftrightarrow \exists$ polynomial $p \in \mathbb{R}[t]\, \forall n \in \mathbb{N}$: $|\mathrm{Sub}(\mathbf{A}^n)| \leq 2^{p(n)}$.

Representation of Clonoids by forks

Lemma (cf., e.g., [Aichinger, 2010])

A finite set, $\mathbf{B}$ finite algebra with Mal'cev term, $\mathbf{C}, \mathbf{D}$ clonoids with source set A and target algebra $\mathbf{B}$. If

1. $\mathbf{C} \subseteq \mathbf{D}$,
2. $\varphi(\mathbf{C}, \mathbf{a}) = \varphi(\mathbf{D}, \mathbf{a})$ for all $\mathbf{a} \in A^*$,

then $\mathbf{C} = \mathbf{D}$.

Lemma [Aichinger and Mayr, 2014]

A finite set, $\mathbf{B}$ finite algebra with k-edge term, $\mathbf{C}, \mathbf{D}$ clonoids with source set A and target algebra $\mathbf{B}$. If

1. $\mathbf{C} \subseteq \mathbf{D}$,
2. $\varphi(\mathbf{C}, \mathbf{a}) = \varphi(\mathbf{D}, \mathbf{a})$ for all $\mathbf{a} \in A^*$,
3. $\mathbf{C}^{[|A|^{k-1}]} = \mathbf{D}^{[|A|^{k-1}]}$,

then $\mathbf{C} = \mathbf{D}$.

Connection between different arities

"Definition"

$\mathbf{v}, \mathbf{w} \in A^*$. $\mathbf{v} \leq_E \mathbf{w} :\Leftrightarrow \mathbf{w}$ can be obtained from $\mathbf{v}$ by inserting letters.
The insertion $\mathbf{xy} \to \mathbf{x}a\mathbf{y}$ is allowed only if a appears in $\mathbf{x}$.

1. $abab \leq_E aabaabb$, since
   $abab \to aabab \to aabaab \to aabaabb = aabaabb$.
2. $aab \not\leq_E abab$.

Theorem [Higman, 1952], [Aichinger et al., 2011]

A finite. $(A^*, \leq_E)$ has no infinite antichains.

Theorem - the connection between forks of different arity [Aichinger and Mayr, 2014]

$\mathbf{C}$ clonoid with source A and target $\mathbf{B}$, $\mathbf{a} \in A^m$, $\mathbf{b} \in A^n$. If $\mathbf{a} \leq_E \mathbf{b}$, then $\varphi(\mathbf{C}, \mathbf{b}) \subseteq \varphi(\mathbf{C}, \mathbf{a})$.

Outline of the order theoretic argument

Let $\mathbf{C}$ be a clonoid with finite source A and finite target algebra $\mathbf{B}$ *with an edge term.* Suppose that we have

$\mathbf{C}_1 \supset \mathbf{C}_2 \supset \cdots$ (descending chain of subclonoids).

From this chain, we construct (using edge terms and the "fork" lemmas)

$U_1 \subset U_2 \subset \cdots$ (ascending chain of upward closed subsets of $(A^*, \leq_E)$).

Now let

$M :=$ minimal elements of $\bigcup U_i$ w.r.t $\leq_E$.

Then M is an infinite antichain in $(A^*, \leq_E)$.

This contradicts Higman's Theorem (its modification by Aichinger/Mayr/McKenzie).

Subvarieties of f.g. varieties

Theorem [Aichinger and Mayr, 2014]

A finite set, $\mathbf{B}$ finite algebra with edge term.
$\mathcal{C} := \{\mathbf{C} \mid \mathbf{C}$ is clonoid with source A and target $\mathbf{B}\}$.
Then $(\mathcal{C}, \subseteq)$ satisfies the (DCC).

Theorem [Aichinger and Mayr, 2014]

$\mathbf{A}$ finite algebra with edge term, $\mathcal{W} :=$ subvarieties of $\mathbb{V}(\mathbf{A})$. Then:

- $(\mathcal{W}, \subseteq)$ satisfies the (ACC).
- Every subvariety of $\mathbb{V}(\mathbf{A})$ is f.g.

Proof: From $W_1 \subset W_2 \subset \cdots$,
we obtain $\mathbf{Th}_{\mathbf{A}}(W_1) \supset \mathbf{Th}_{\mathbf{A}}(W_2) \supset \cdots$,
which is an infinite descending chains of clonoids with source A and target $\mathbf{B} := \mathbf{A} \times \mathbf{A}$. Contradiction.

(DCC) for subvarieties

Theorem [Aichinger and Mayr, 2014]

$\mathbf{A}$ finite algebra with edge term. Then every subvariety of $\mathbb{V}(\mathbf{A})$ is f.g.

Corollary [Aichinger and Mayr, 2014]

$\mathbf{A}$ finite algebra with an edge term. Then the following are equivalent:

1. There is no infinite descending chain of subvarieties of $\mathbb{V}(\mathbf{A})$.
2. Each $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ is finitely based relative to $\mathbb{V}(\mathbf{A})$.
3. $\mathbb{V}(\mathbf{A})$ has only finitely many subvarieties.
4. $\mathbb{V}(\mathbf{A})$ contains, up to isomorphism, only finitely many cardinality critical members.

$\mathbf{B}$ is *cardinality critical* $:\Leftrightarrow \mathbf{B} \notin \mathbb{V}(\{\mathbf{C} \mid \mathbf{C} \in \mathbb{V}(\mathbf{B}), |\mathbf{C}| < |\mathbf{B}|\})$.

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