Polynomial interpolation in algebras with a Mal'cev term

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Interpolation

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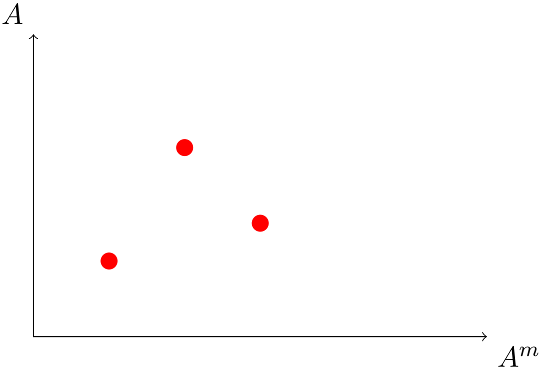

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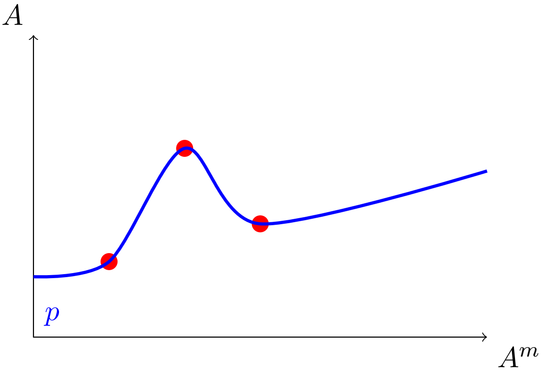

Polynomial interpolation: describe sets of points (partial functions) that can be interpolated by a term function or polynomial function.

Relational description of interpolable functions

- Let C be a clone on A.
- Which $f : D \to A$ with $D \subseteq A^m$ can be interpolated at every subset $T \subseteq D$ with $|T| \leq k$ by a function in C?
- Answer: Exactly the functions that preserve all k-ary invariant relations of C:

$$k\text{-place interpolable partial functions} = \boldsymbol{pPol}\ \boldsymbol{Inv}^{[k]}\ C.$$

- Reason: If f preserves $\rho = \{c|_T : c \in C\} \subseteq A^T$, then $f|_T = f(\pi_1^{(m)}|_T, \ldots, \pi_m^{(m)}|_T) \in \rho$, and so there is $c \in C$ with $f|_T = c|_T$.

The relational degree

Let C be a clone on A.

- The **total relational degree** of C is defined by

$$\deg_{\text{tot}}(C) := \inf \{k \mid \textbf{\textit{Pol Inv}}^{[k]}\, C = C\}.$$

Hence: $\deg_{\text{tot}}(C) \leq k$ if and only if every k-place interpolable function is in C.

- The **partial relational degree** of C is defined by

$$\deg_{\text{part}}(C) := \inf \left\{k \mid \textbf{\textit{pPol Inv}}^{[k]}\, C = \{c|_T \ :\ m \in \mathbb{N}, c \in C^{[m]}, T \subseteq A^m\}\right\}.$$

Hence: $\deg_{\text{part}}(C) \leq k \iff$ every k-place interpolable partial function is a restriction of a function in C.

Relational degree

The relational degree

Partial interpolation $\neq$ total interpolation:

Let $\mathbf{A} = (\mathbb{Z}_2, +, 1)$.

1. Then $\operatorname{Clo}\mathbf{A} = \boldsymbol{Pol}(\{(a,b,c,d) \in A^4 \mid a+b=c+d\})$, hence $\deg_{\text{tot}}(\operatorname{Clo}\mathbf{A}) \leq 4$.
2. $\deg_{\text{part}}(\operatorname{Clo}\mathbf{A}) = \aleph_0$:
   For $k \geq 2$, the $(k+1)$-ary near-unanimity operation preserves every k-ary relation, but cannot be interpolated by a function in $\operatorname{Clo}(\mathbf{A})$.

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Polynomial interpolation $\neq$ term interpolation:

Let $\mathbf{A} := (\{0,1\}, x \wedge (y \vee z))$.

1. $\deg_{\mathrm{tot}}(\operatorname{Clo}\mathbf{A}) = \aleph_0$ because $\mathbf{A}$ is not finitely related.
2. $\deg_{\mathrm{tot}}(\operatorname{Pol}\mathbf{A}) = 2$ because $\operatorname{Pol}\mathbf{A} = \operatorname{Pol}(\{0,1\}, \vee, \wedge)$ and every monotone function is polynomial.

Results on the total relational degree

Let $\mathbf{A}$, $\mathbf{B}$ be finite algebras in the same language. Then we have

- $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{B})$ and $\deg_{\text{tot}}(\operatorname{Clo}\mathbf{A}) < \aleph_0 \implies \deg_{\text{tot}}(\operatorname{Clo}\mathbf{B}) < \aleph_0$.
- There are finite groupoids $\mathbf{A}, \mathbf{B}$ with $\deg_{\text{tot}}(\operatorname{Clo}\mathbf{A}) < \aleph_0$, $\mathbf{B} \in \mathbb{V}(\mathbf{A})$, $\deg_{\text{tot}}(\operatorname{Clo}\mathbf{B}) = \aleph_0$.

  [Davey, Jackson, Pitkethly, C.Szabó 2011]
- $\mathbf{A}$ has an edge term $\implies \deg_{\text{tot}}(\operatorname{Clo}\mathbf{A}) < \aleph_0$.

  [Aichinger, Mayr, McKenzie 2014]
- $\mathbf{A}$ is in a cm variety, $\deg_{\text{tot}}(\operatorname{Clo}\mathbf{A}) < \aleph_0 \implies \mathbf{A}$ has an edge term.

  [Barto 2018]
- $\mathbf{A}$ of finite type, with edge term, $\mathbb{V}(\mathbf{A})$ residually small $\implies$ $\exists k \in \mathbb{N}\ \forall \mathbf{C} \in \mathbb{V}(\mathbf{A}) : \deg_{\text{tot}}(\operatorname{Clo}\mathbf{C}) \leq k$.

  Combine the above. Computable k by [Kearnes, Szendrei 2012]

Results on the partial relational degree

Theorem (Baker-Pixley 1975)

Let C be a clone on a finite set A, and let $k \geq 2$. Then

$$\deg_{\text{part}}(C) \leq k \iff C \text{ has a } (k+1)\text{-ary near unanimity function.}$$

Proof of $\Longrightarrow$: Let $u : \bigcup_{i \in \underline{k+1}} \{(x, \ldots, x, \underset{i}{y}, x, \ldots, x) \mid x, y \in A\} \to A$ be the $(k+1)$-ary nu operation, let $R \subseteq A^k$.

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$$M := \begin{pmatrix} x_1 & y_1 & x_1 & \ldots & x_1 \\ y_2 & x_2 & x_2 & \ldots & x_2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_k & x_k & x_k & \ldots & y_k \end{pmatrix}$$

be such that the rows of M are in $\operatorname{dom}(u)$ and the columns are in R.

Results on the partial relational degree

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Proof of $\Longrightarrow$: Let $u : \bigcup_{i \in \underline{k+1}} \{(x, \ldots, x, \underset{i}{y}, x, \ldots, x) \mid x, y \in A\} \to A$ be the $(k+1)$-ary nu operation, let $R \subseteq A^k$. We need to show:

$$\begin{pmatrix} u(x_1, y_1, x_1, \ldots, x_1) \\ u(y_2, x_2, x_2, \ldots, x_2) \\ \vdots \\ u(x_k, x_k, x_k, \ldots, y_k) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in R.$$

There are k rows and $k+1$ columns, hence there is a column without y. Hence by assumption $(x_1, \ldots, x_k) \in R$.

Baker-Pixley and polynomial interpolation

Theorem (Baker-Pixley 1975)

Let C be a clone on a finite set A, and let $k \geq 2$. Then
$\deg_{\text{part}}(C) \leq k \iff C$ has a $(k+1)$-ary near unanimity function.

Consequence for polynomial interpolation in cp varieties:
If $\mathbf{A}$ is a finite algebra with a Mal'cev term,

$$\deg_{\text{part}}(\operatorname{Pol}\mathbf{A}) \in \mathbb{N}_0, \;\; k \geq \deg_{\text{part}}(\operatorname{Pol}\mathbf{A}) \;\text{ and }\; k \geq 2,$$

then

- $\mathbf{A}$ has a $(k+1)$-ary near-unanimity polynomial,
- $\mathbb{HSP}\,\mathbf{A}^*$ is arithmetical (cd + cp),
- $\deg_{\text{part}}(\operatorname{Pol}\mathbf{A}) \in \{0, 2\}$ and $\mathbf{A}$ is strictly affine complete (every partial congruence preserving function with finite domain is polynomial).

Polynomial completeness properties for partial functions

Collapse of completeness conditions

Theorem. $\mathbf{A}$ ... finite algebra in a cp variety, $k \in \mathbb{N}$, $\mathcal{R} \subseteq \mathcal{P}(A^k)$.
If every partial $\mathcal{R}$-preserving function is polynomial, then $\mathbf{A}$ is strictly affine complete.

Hence: If

$$\boldsymbol{pPol}(\mathcal{R}) = \{p|_T : n \in \mathbb{N},\, p \in \operatorname{Pol}^{[n]} \mathbf{A},\, T \subseteq A^n\} = \boldsymbol{pPol}\ \boldsymbol{Inv} \operatorname{Pol} \mathbf{A},$$

then

$$\boldsymbol{pPol}(\operatorname{Con}(\mathbf{A})) = \{p|_T : n \in \mathbb{N},\, p \in \operatorname{Pol}^{[n]} \mathbf{A},\, T \subseteq A^n\}.$$

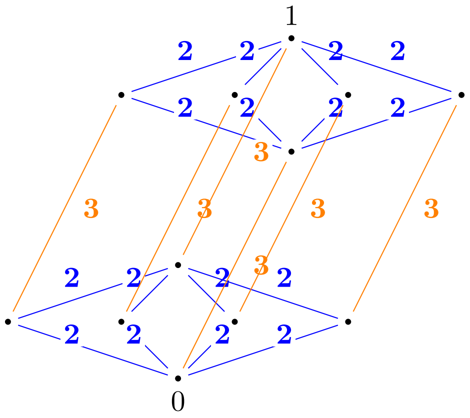
Polynomially rich algebras

Let $\mathbf{A}$ be a finite algebra in a cp variety, and let $f \in \operatorname{Pol} \mathbf{A}$. Then

- f preserves all congruences of $\mathbf{A}$.
- Hence $\operatorname{Con}(\mathbf{A}) = \operatorname{Con}(\mathbf{A} + f)$.
- f also preserves the labelling of the congruence lattice according to Tame Congruence Theory [Hobby McKenzie 1988] .

The labelled congruence lattice

$\mathrm{Con}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times A_5)$



Polynomially rich algebras

Let $\mathbf{A}$ be in a cp variety.

- The labelling is completely determined by certain 4-ary relations encoding $[\beta, \beta] \leq \alpha$.
- These relations are $\{(a, b, c, d) \in A^4 \mid a \,\beta\, b \,\beta\, c,\ m(a, b, c) \,\alpha\, d\}$.
- $\mathbf{A}$ is called polynomially rich if every congruence preserving function that preserves these 4-ary relations is polynomial. [Idziak and Słomczyńska 2001] .
- We call $\mathbf{A}$ strictly polynomially rich if every partial function preserving ... with finite domain is polynomial.
- But then we have:

  $\mathbf{A}$ strictly polynomially rich $\Leftrightarrow$ $\mathbf{A}$ is strictly affine complete $\Leftrightarrow$ $\mathbf{A}$ has a Pixley polynomial.

Strictly polynomially rich algebras

Let $\mathcal{R}$ be the set of congruences and the 4-ary relations encoding the labelling of the congruence lattice.

- If every 5-ary $\mathcal{R}$-preserving partial function is polynomial, then **A** has a 5-ary near-unanimity polynomial, and is thus strictly affine complete.
- But what if only every 4-ary $\mathcal{R}$-preserving function is polynomial? Or just every 1-ary partial $\mathcal{R}$-preserving function?

Definition

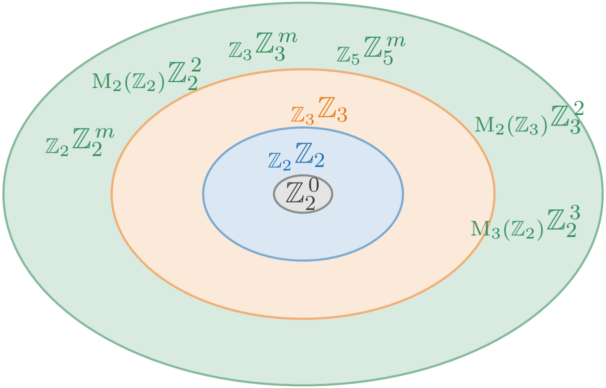
A is **strictly k-polynomially rich** if every partial k-ary function preserving congruences and the 4-ary "labelling relations" with finite domain is polynomial.

Goal: Characterize strictly k-polynomially rich algebras for $k \leq 4$.

Strictly polynomially rich algebras

Finite modules over simple rings

Strictly k-polynomially rich modules over the matrix rings $M_n(\mathbf{D})$ have been characterized by M. Kapl in 2005:



strictly 1-polynomially rich
strictly 2-polynomially rich
strictly 3-polynomially rich
strictly 4-polynomially rich

${}_{\mathbb{Z}_7}\mathbb{Z}_7^2$

Strictly polynomially rich Mal'cev algebras

Theorem [Rossi 2024]

$\mathbf{A}$ finite algebra in cp variety. TFAE:

1. Every $\mathbf{B} \in \mathbb{H}\mathbf{A}$ is strictly 2-polynomially rich.
2. - There is a chain $0 = \alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_n = 1$ in $\mathrm{Con}(\mathbf{A})$ such that all α_i are distributive elements of $\mathrm{Con}(\mathbf{A})$ and for each i, $\mathtt{typ}(\alpha_{i-1}, \alpha_i) = \mathbf{3}$ or ($\mathtt{typ}(\alpha_{i-1}, \alpha_i) = \mathbf{2}$ and every α_i-class is the union of at most three α_{i-1}-classes).
   - $[\gamma, \delta] = \gamma \cap \delta$ for all join irreducible γ, δ with $\gamma \neq \delta$. (SC1)

Corollary. The group S_3 is strictly 2-polynomially rich.

Strictly polynomially rich Mal'cev algebras

Theorem [Rossi 2024]

$\mathbf{A}$ finite algebra in cp variety. TFAE:

1. Every $\mathbf{B} \in \mathbb{H}\mathbf{A}$ is strictly 4-polynomially rich.
2. $\mathbf{A}$ is strictly affine complete.
3. $\mathbf{A}$ has a nu-polynomial.

Improvement: 4 instead of 5.

Details: [E. Aichinger, M. Kapl, B. Rossi, *Polynomial interpolation of partial functions in finite algebras with a Mal'cev term*, arXiv March 2026]