FINITE REPRESENTATION OF HIGHER COMMUTATORS

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Higher Commutators

Higher Commutators:

- Higher commutators were introduced (as multi-placed commutators) in 2001 by A. Bulatov to distinguish the clones

$$\begin{aligned} \mathrm{Pol}(\mathbf{B}_2) &= \mathrm{Pol}\left(\mathbb{Z}_4, +, 2x_1x_2\right) \text{ and} \\ \mathrm{Pol}(\mathbf{B}_3) &= \mathrm{Pol}\left(\mathbb{Z}_4, +, 2x_1x_2x_3\right). \end{aligned}$$

- These clones differ in their 8-ary invariant relations.
- The top congruence 1 satisfies $[1,1,1]_{\mathbf{B}_2} = 0$ and $[1,1,1]_{\mathbf{B}_3} > 0$.

What are higher commutators?

Let $\mathbf{A}$ be an algebra, $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathrm{Con}(\mathbf{A})$. Then

$$[\alpha_1, \ldots, \alpha_n]_{\mathbf{A}}$$

is an element of $\mathrm{Con}(\mathbf{A})$. It is called the higher commutator of $\alpha_1, \ldots, \alpha_n$.

Lemma. Let $\mathbf{V} = (V, +, -, 0, (f_i)_{i \in I})$ be an expanded group, and let $A_1, \ldots, A_n$ be ideals of $\mathbf{V}$.
Then $[A_1, \ldots, A_n]_{\mathbf{V}}$ is the ideal generated by

$$\{p(a_1, \ldots, a_n) \mid p \in \mathrm{Pol}_n(\mathbf{V}), a_1 \in A_1, \ldots, a_n \in A_n, \\ \forall x_1, \ldots, x_n \in V : 0 \in \{x_1, \ldots, x_n\} \Rightarrow p(x_1, \ldots, x_n) = 0\}.$$

Definitions of higher commutators

- Bulatov's definition applies to every algebraic structure (A, F).
- Different types of higher commutators, the two term higher commutator and the hypercommutator, have been introduced by A. Moorhead in 2018 and 2021.

Applications of Higher Commutators

Definition. $\mathbf{A}$ is supernilpotent if $\exists k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall \alpha_1, \ldots, \alpha_n \in \mathrm{Con}(\mathbf{A})$:

$$n > k \Rightarrow [\alpha_1, \ldots, \alpha_n]_{\mathbf{A}} = 0_A.$$

Theorem [G. Higman (1965) – Berman, Blok, Freese, Hobby, McKenzie – A. Wires (2019)]
Let $\mathbf{A}$ be a finite algebra in a cm variety. Then $\mathbf{A}$ is supernilpotent if and only if

$$\exists p \in \mathbb{R}[x]\ \forall n \in \mathbb{N}: \ |\mathrm{Clo}_n(\mathbf{A})| \leq 2^{p(n)}.$$

Theorem [Kearnes, Rasstrigin (2020)]
If $\mathbf{A}$ is two term supernilpotent, then every subalgebra of $\mathbf{A}$ is a homomorphic image of a finite subdirect power of $\mathbf{A}$.

The sequence of higher commutator operations

Let $\mathbf{A}$ be an algebra with congruence lattice $\mathbb{L}$. The higher commutator operations of $\mathbf{A}$ can be collected into the function

$$\begin{aligned} C_{\mathbf{A}} \quad : \quad (\textstyle\bigcup_{n\in\mathbb{N}} \mathbb{L}^n) &\rightarrow \mathbb{L}, \\ C_{\mathbf{A}}(\alpha_1, \dots, \alpha_n) &= [\alpha_1, \dots, \alpha_n]_{\mathbf{A}}. \end{aligned}$$

First goal:

- Finite representation of $C_{\mathbf{A}}$ for finite $\mathbf{A}$ (or $\mathbb{L}$).

The sequence of higher commutator operations

We want to represent

$$\begin{array}{rcl} C_{\mathbf{A}} \quad : \quad (\bigcup_{n\in\mathbb{N}} \mathbb{L}^n) & \to & \mathbb{L}, \\ C_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n) & = & [\alpha_1, \ldots, \alpha_n]_{\mathbf{A}}. \end{array}$$

Theorem

- $C_{\mathbf{A}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq C_{\mathbf{A}}(\alpha_2, \ldots, \alpha_n)$ (omission property).
- $C_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n) = C_{\mathbf{A}}(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$ for all $\pi \in S_n$ with $\pi(n) = n$.
- $\mathbf{A}$ in a cm variety $\Rightarrow C_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n) = C_{\mathbf{A}}(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$ for all $\pi \in S_n$. (symmetry, [Moorhead 2018]).

Encoding

We call $\mathbf{A}$ hc-symmetric if $C_{\mathbf{A}}(\alpha_1, \dots, \alpha_n) = C_{\mathbf{A}}(\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})$ for all $n \in \mathbb{N}$, $\pi \in S_n$.

For hc-symmetric finite $\mathbf{A}$ and $\mathbb{L} = \mathrm{Con}(\mathbf{A}) = \{\lambda_1, \dots, \lambda_m\}$, we can compute

$$C_{\mathbf{A}}(\alpha_1, \dots, \alpha_n) = C_{\mathbf{A}}(\underbrace{\lambda_1, \dots, \lambda_1}_{a_1}, \dots, \underbrace{\lambda_m, \dots, \lambda_m}_{a_m}),$$

where a_j is the number of occurrences of λ_j in $(\alpha_1, \dots, \alpha_n)$.

Definition. $F : {\mathbb{N}_0}^m \to \mathbb{L}$ with

$$F(a_1, \dots, a_m) := C_{\mathbf{A}}(\underbrace{\lambda_1, \dots, \lambda_1}_{a_1}, \dots, \underbrace{\lambda_m, \dots, \lambda_m}_{a_m})$$

is the encoding of $C_{\mathbf{A}}$.

Encoding as an antitone function

The omission property

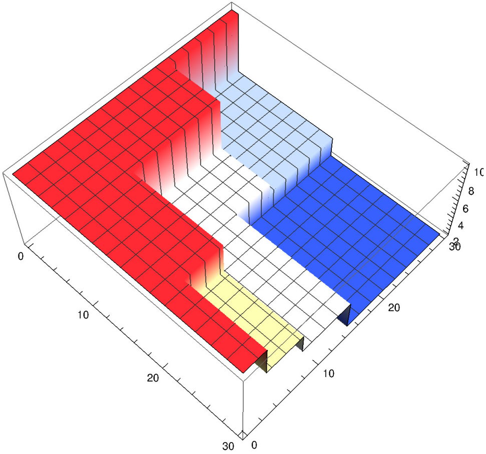
$$C_{\mathbf{A}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq C_{\mathbf{A}}(\alpha_2, \ldots, \alpha_n)$$

and hc-symmetry imply:

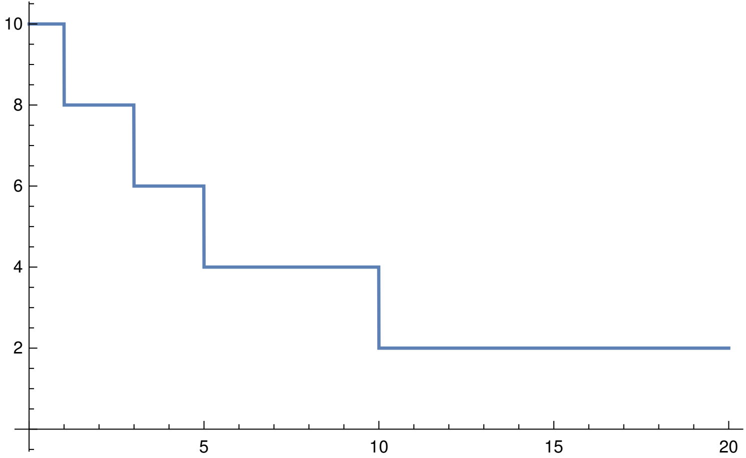
$$(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m) \Rightarrow F(a_1, \ldots, a_m) \geq F(b_1, \ldots, b_m).$$

Theorem. The encoding of $C_{\mathbf{A}}$ of a finite hc-symmetric algebra $\mathbf{A}$ is an antitone function from ${\mathbb{N}_0}^m$ to $\mathbb{L} = \mathrm{Con}(\mathbf{A})$.

An antitone function from ${\mathbb{N}_0}^2$ to $\{1, \ldots, 10\}$



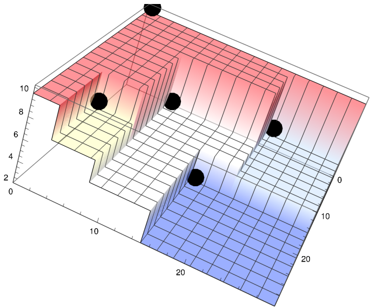
An antitone function from $\mathbb{N}_0$ to $\{1, \ldots, 10\}$

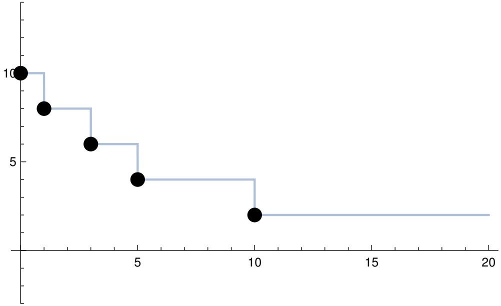


Fact on antitone functions

Let $f : \mathbb{N}_0{}^m \to \mathbb{L}$ be antitone ($\mathbb{L}$ finite). Then there is a finite set $G \subseteq \mathbb{N}_0{}^m \times \mathbb{L}$ such that

$$f(\boldsymbol{x}) = \bigwedge \{\alpha \mid (\boldsymbol{a}, \alpha) \in G, \boldsymbol{a} \leq \boldsymbol{x}\}.$$





Consequence for Commutators

- $\mathbf{B} = (\mathbb{Z}_4, +, (2x_1 \cdots x_n)_{n\in\mathbb{N}})$, $\mathrm{Con}(\mathbf{B}) = \{0, \alpha, 1\}$. Then $C_{\mathbf{B}}$ is the largest function with symmetry and the omission property such that

$$[1,1] = \alpha,\ [1,\alpha] = 0,\ [\alpha,\alpha] = 0, [\alpha] = \alpha, [0] = 0.$$

- $\mathbf{B}_3 = (\mathbb{Z}_4, +, 2x_1x_2x_3)$. Then $C_{\mathbf{B}_3}$ is the largest function with symmetry and the omission property such that

$$[1,1,1,1] = 0,\ [1,1] = \alpha,\ [1,\alpha] = 0,\ [\alpha,\alpha] = 0, [\alpha] = \alpha, [0] = 0.$$

From these finite lists of commutator equalities, we can compute all higher commutators.

Representing $C_{\mathbf{A}}$ by commutator equalities

Theorem. Let $\mathbf{A}$ be a finite algebra in a cm variety. Then there is a finite set Φ of commutator equalities of the form

$$[\alpha_1, \ldots, \alpha_n] = \beta$$

such that the higher commutator function $C_{\mathbf{A}}$ is the largest function with symmetry and the omission property that satisfies Φ.

From Φ, we can then compute all $[\gamma_1, \ldots, \gamma_k]$.

Question: Is Φ computable from $(A, (f_i)_{i \in I})$?

Computing higher commutators

On input $\mathbf{A} = (A, (f_i)_{i \in I})$ (finite, finite type), we can compute

- $[\alpha_1, \ldots, \alpha_n]$ for each given tuple $\bar{\alpha}$ (almost by Bulatov's definition).
- If $\mathbf{A}$ lies in a cm variety, we can decide whether

  $$[1,1] \cap [1,1,1] \cap [1,1,1,1] \cap \cdots = 0,$$

  i.e., whether $\mathbf{A}$ is supernilpotent, and hence we can compute $[1,1] \cap [1,1,1] \cap [1,1,1,1] \cap \cdots$.
- If $\mathbf{A}$ lies in a cm variety, then given α, we can compute

  $$[\alpha,\alpha] \cap [\alpha,\alpha,\alpha] \cap [\alpha,\alpha,\alpha,\alpha] \cap \cdots .$$

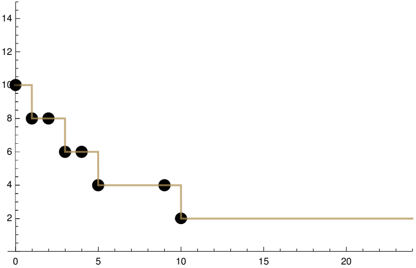
  [Mayr, Á. Szendrei 2021].

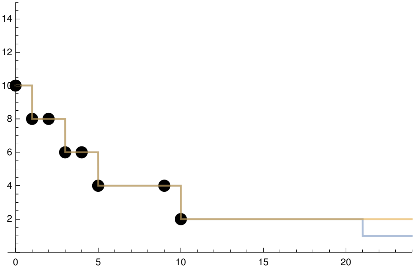
Computing higher commutators

- We do not know whether we can compute $[\alpha, \beta] \cap [\alpha, \alpha, \beta, \beta] \cap [\alpha, \alpha, \alpha, \beta, \beta, \beta] \cap \cdots$.

Determining C_{A} uniquely

Observation. There is no finite set of points P such that





is the only antitone function interpolating P.

Determining Higher Commutators Uniquely

Proposition. For every finite set Ψ and for every sequence with

$$[1, 1, \ldots, 1]_n > 0 \text{ for all } n \in \mathbb{N},$$

the sequence is not determined uniquely by Ψ among all sequences with symmetry and the omission property.

Solution. Allow extended equalities of the form

$$[S; \alpha_1, \ldots, \alpha_n] = \beta,$$

where

$$[S; \alpha_1, \ldots, \alpha_n] := \bigwedge_{k \in \mathbb{N}_0} \bigwedge_{(\sigma_1, \ldots, \sigma_k) \in S^k} [\sigma_1, \ldots, \sigma_k, \alpha_1, \ldots, \alpha_n].$$

Determining Higher Commutators Uniquely

- $\mathbf{B} = (\mathbb{Z}_4, +, (2x_1 \cdots x_n)_{n \in \mathbb{N}})$, $\mathrm{Con}(\mathbf{B}) = \{0, \alpha, 1\}$. Then $C_{\mathbf{B}}$ is the unique function with symmetry and the omission property such that

$$[1,1] = \alpha, [1,\alpha] = 0, [\alpha,\alpha] = 0, [0] = 0, [\alpha] = \alpha,$$
$$[1] = 1, [\{1\};1,1] = \alpha, [\{0,\alpha,1\};1] = 0,$$

- Some equalities are redundant if we know more commutator properties.
- $[\{1\};1,1] = \alpha$ excludes $[1,\ldots,1]_{99} = \alpha$, $[1,\ldots,1]_{100} = 0$.

Determining Higher Commutators Uniquely

Theorem. Let $\mathbf{A}$ be a finite algebra in a cm variety. Then there is a finite set Ψ of extended commutator equalities such that the higher commutator operations $C_{\mathbf{A}}$ of $\mathbf{A}$ are the unique sequence satisfying Ψ, symmetry and the omission property.

From Ψ, we can effectively evaluate all $[\alpha_1, \ldots, \alpha_n]$ and $[S; \alpha_1, \ldots, \alpha_n]$.

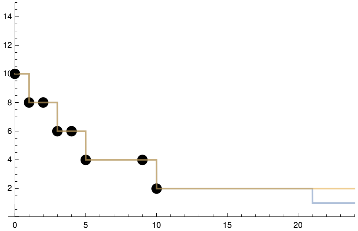
Extensions of antitone functions

- We represented $C_{\mathbf{A}}$ by an antitone function $F : {\mathbb{N}_0}^m \to \mathbb{L}$.
- Its extension $\widehat{F} : (\mathbb{N}_0 \cup \{\infty\})^m \to \mathbb{L}$ is defined by

$$\widehat{F}(\boldsymbol{x}) := \bigwedge \{F(\boldsymbol{b}) \mid \boldsymbol{b} \le \boldsymbol{x}, \boldsymbol{b} \in {\mathbb{N}_0}^m\}.$$

  This is the continuous extension of F to $(\mathbb{N}_0 \cup \{\infty\})^m$, seen as the m-fold product of the Alexandroff extension of $\mathbb{N}_0$ with the discrete topology.
- Continuous antitone functions can be represented by finite subsets of their graph.

Unique representation of antitone functions



The upper function is uniquely determined among antitone functions by

$$\text{black points and } \widehat{f}(\infty) = 2,$$

the lower function by

$$\text{black points and } f(20) = 2 \text{ and } f(21) = 1 \text{ and } \widehat{f}(\infty) = 1.$$

Distinguishing higher commutator operations

Problem. We are given two finite algebras $\mathbf{A}$ and $\mathbf{B}$ in cm varieties with the same congruence lattice. Task: Determine whether they have the same higher commutator operations.

- After verifying a finite number of $[\alpha_1, \ldots, \alpha_n]_{\mathbf{A}} = [\alpha_1, \ldots, \alpha_n]_{\mathbf{B}}$, we may still have $C_{\mathbf{A}} \neq C_{\mathbf{B}}$.
- We can determine $C_{\mathbf{A}} = C_{\mathbf{B}}$, provided we can solve one of the following tasks:
  - ☐ Find sets of commutator equalities $\Phi_{\mathbf{A}}, \Phi_{\mathbf{B}}$ such that $C_{\mathbf{A}}$ is largest with $\Phi_{\mathbf{A}}$ + symmetry + omission property, similar for $C_{\mathbf{B}}$ and $\Phi_{\mathbf{B}}$.
  - ☐ Find a sets of extended commutator equalities $\Psi_{\mathbf{A}}$, $\Psi_{\mathbf{B}}$ such that they uniquely determine $C_{\mathbf{A}}$ and $C_{\mathbf{B}}$.
  - ☐ Evaluate $[S; \alpha_1, \ldots, \alpha_n]_{\mathbf{A}}$ for $S \subseteq \mathrm{Con}(\mathbf{A})$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Con}(\mathbf{A})$.

Learning higher commutator operations

We have an algorithm that finds a set of extended commutator equalities $\Psi_{\mathbf{A}}$ by evaluating finitely many expressions of the form $[S; \alpha_1, \ldots, \alpha_n]_{\mathbf{A}}$.

Definition. A class $\mathcal{V}$ of algebras has computable extended commutator sequences if there is an algorithm which, given $\mathbf{A} = (A, (f_i)_{i \in N}) \in \mathcal{V}$ (by the operation tables of all f_i), $S \subseteq \mathrm{Con}(\mathbf{A})$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Con}(\mathbf{A})$, computes the extended commutator $[S; \alpha_1, \ldots, \alpha_n]$.

Open Question.
Which hc-symmetric classes of algebras have computable extended commutator sequences?