How to express conjunction with few variables

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Boolean conjunction

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$$\begin{aligned} f(0,0) &= 0 \\ f(0,1) &= 0 \\ f(1,0) &= 0 \\ f(1,1) &= 1 \end{aligned}$$

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we will use this notation

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Then $x \vee y = \neg(\neg(x) \wedge \neg(y))$.

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The answer is known and will be presented it in the next slides...

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Before we consider some more examples...

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Example Let $f \colon \{0,1\}^n \to \{0,1\}$ be

$$f(a_1,\ldots,a_n) = \begin{cases} 0 & \text{if } 0 \in \{a_1,\ldots,a_n\} \\ 1 & \text{otherwise.} \end{cases}$$

Then

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When do we only need two variable symbols to represent $\wedge$?

Structure of the talk

Clones and Post's Lattice

Algebras, terms and polynomials

Short conjunctions

Applications to the study of the complexity of the polynomial satisfiability problem

Clones and Post's Lattice

Function composition

Let A be a set, let $f \in A^{A^n}$, let $g_1, \ldots, g_n \in A^{A^k}$.
The composition of f with $g_1, \ldots, g_n$ is the element of A^{A^k}

$$(f \circ (g_1, \ldots, g_n))(a_1, \ldots, a_k) = f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k)).$$

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Example

Let $f = x_1^2 + x_2^2 \in \mathbb{R}^{\mathbb{R}^2}$
let $g_1 = \cos(x_1) \in \mathbb{R}^{\mathbb{R}^1}$
let $g_2 = \sin(x_1) \in \mathbb{R}^{\mathbb{R}^1}$. Then

$$(f \circ (g_1, g_2)) = \cos^2(x_1) + \sin^2(x_1) = 1 \in \mathbb{R}^{\mathbb{R}^1}.$$

Projections

Let A be a set and let $k, n \in \mathbb{N}$ with $k \leq n$.
The k-th n-ary projection is the function

$$\pi_k^n(a_1, \ldots, a_n) = a_k.$$

Clones

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The functions induced by words of the form

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Example Let $\mathbf{K}$ be a field. The functions induced on K by the elements of $\mathbf{K}[x_1, \dots, x_n]$ (i.e. the polynimial functions) form a clone on K.

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Theorem

The set $\mathcal{L}_A$ of the clones on a set A is a complete lattice with respect to set inclusion with top element $\mathcal{O}_A$ and bottom element $\mathcal{J}_A$.

Generating clones

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Example Let $A = \{0, 1\}$. The clone generated by $\wedge, \vee, 0, 1$ is the clone of monotone operations.

Clones on finite sets

Theorem [Ágoston, Demetrovics, Hánnak, 1983]

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Theorem [Post, 1944]

On the two-element set there are

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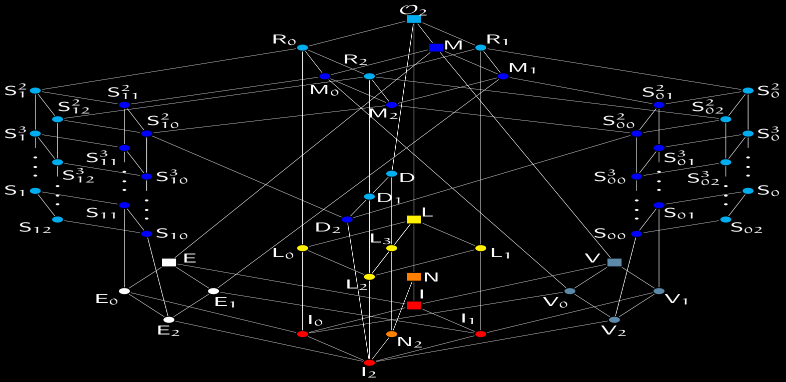
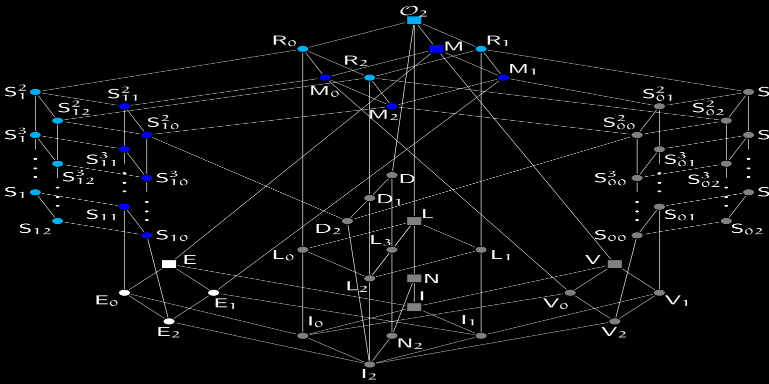
The lattice of clones on the two-element set is called **Post's Lattice**

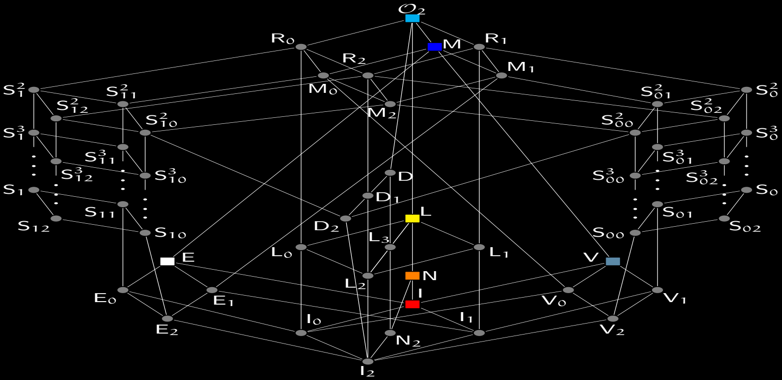
Post's Lattice



Clones that contain the Boolean conjunction



Constantive clones on $\{0, 1\}$



Algebras, terms and polynomials

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The operation $f_0^{\mathbf{G}} \colon \{\emptyset\} \to G$ is defined by

$$0.$$

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The term operations form a clone
$\mathrm{Clo}\,\mathbf{G}$ = the clone generated by the basic operations of $\mathbf{G}$.

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The map f does not:

$$0 \leq 1 \text{ and } f(0) = 1 \not\leq 0 = f(1).$$

Hence f cannot be a composition of maps that preserve $\leq$.

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How can we formally define the number of variables needed to represent a term operation?

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From terms to term operations

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We have that

$$\operatorname{Clo} \mathbf{A} = \{\, t^{\mathbf{A}} \mid t \in T_L(n) \,\}.$$

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$t \in T_L(n)$ is frugal if t contains exactly one occurrence of each symbol $x_1, \dots, x_n$.

Short conjunctions

Let $\mathbf{A} = (\{0, 1\}; L)$. Is there a frugal $t \in T_L(2)$ such that $t^{\mathbf{A}}$ is the Boolean conjunction?

Polynomials

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The polynomial operations of $\mathbf{A}$ are the operations induced by the elements of $T_{\mathbf{A}}(\omega)$. They form a clone $\mathrm{Pol}\,\mathbf{A}$.

Polynomials and polynomial operations

The map $f\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2$ given by

$$(x_1, x_2) \mapsto 1 + x_1 + 0 - x_2 + x_1 + 0$$

is a polynomial operation of the group $\mathbf{G} = (\mathbb{Z}_2; +, -, 0)$.

Short conjunctions

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Theorem [Aichinger, R.]

Let $\mathbf{A} = (\{0,1\}; \mathrm{L})$ and let us assume that $\wedge \in \operatorname{Pol}\mathbf{A}$. Then there exists a frugal $\mathrm{t} \in \mathrm{T}_{\mathbf{A}}(2)$ such that $\mathrm{t}^{\mathbf{A}} = \wedge$.

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Let $\mathbf{A} = \mathrm{GF}(2)$ and let $p \in T_{\mathbf{A}}(4) = \mathrm{GF}(2)[x_1, x_2, x_3, x_4]$ be

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$$\begin{aligned}\wedge \in \mathrm{Pol}\,\mathbf{A} \Rightarrow &\exists m \in \mathbb{N} \\ &\exists s \in T_{\mathbf{A}}(m) \text{ frugal} \\ &\exists \tau\colon \{1,\ldots,m\} \to \{1,2\} \\ &\forall a_1, a_2 \in \{0,1\}:\end{aligned}$$

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$$a_1 \wedge a_2 = s^{\mathbf{A}}(a_{\tau(1)},\dots,a_{\tau(m)}) = \left(s^{\mathbf{T_A}(2)}(x_{\tau(1)},\dots,x_{\tau(m)})\right)^{\mathbf{A}}(a_1,a_2)$$

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Let $m = 5$, and $\tau = \{(1,1),(2,2),(3,2),(4,1),(5,1)\}$. Then

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We construct $\tilde{s} \in T_{\mathbf{A}}(\tilde{m})$ frugal and $\tilde{\tau}\colon \{1,\ldots,\tilde{m}\} \to \{1,2\}$ with

$$\begin{aligned} &\tilde{m} < m, \\ &\wedge = \left(\tilde{s}^{\mathbf{T}_{\mathbf{A}}(2)}(x_{\tilde{\tau}(1)},\ldots,x_{\tilde{\tau}(\tilde{m})})\right)^{\mathbf{A}} \end{aligned}$$

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$$\exists w \in \mathsf{T}_{\mathbf{A}}(1)\colon w \text{ is frugal, } w^{\mathbf{A}}(0) = 1,\ w^{\mathbf{A}}(1) = 0$$

Case 1.1: $s^{\mathbf{A}}(0, 1, 1, 0, 1) = 1$ and $s^{\mathbf{A}}(0, 0, 0, 0, 1) = 0$:

$$\wedge = \left(s^{\mathbf{T}_{\mathbf{A}}(2)}(0, x_1, x_1, \boxed{0}, x_2)\right)^{\mathbf{A}}$$

last but one 0 of $(0, 1, 1, 0, 0)$

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Case 1.2: $s^{\mathbf{A}}(0,1,1,0,1) = 0$ and $s^{\mathbf{A}}(0,0,0,0,1) = 1$:

$$\wedge = \left(s^{\mathbf{T}_{\mathbf{A}}(2)}(0, w^{\mathbf{T}_{\mathbf{A}}(1)}(x_1), w^{\mathbf{T}_{\mathbf{A}}(1)}(x_1), 0, x_2)\right)^{\mathbf{A}}$$

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Case 1.4: $s^{\mathbf{A}}(0,1,1,0,1) = s^{\mathbf{A}}(0,0,0,0,1) = 1$:

$$\wedge = \left(w^{\mathbf{T_A}(2)}\left(s^{\mathbf{T_A}(2)}(x_2, w^{\mathbf{T_A}(1)}(x_1), w^{\mathbf{T_A}(1)}(x_1), x_2, 1)\right)\right)^{\mathbf{A}}$$

Proof by example

Case 2: The negation is not induced by a frugal polynomial.

Case 2.1: $s^{\mathbf{A}}(0,1,1,1,1) = 1$:

$$\wedge = \left(s^{\mathbf{T}_{\mathbf{A}}(2)}(\,0\,,\ x_2, x_2\,,\ x_1, x_1\,)\right)^{\mathbf{A}}.$$

Proof by example

Case 2: The negation is not induced by a frugal polynomial.

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0s of $(0,1,1,0,0)$

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In fact $s^{\mathbf{A}}(0,0,0,1,1) = 0$. Since, if $s^{\mathbf{A}}(0,0,0,1,1) = 1$, then $s^{\mathbf{T}_{\mathbf{A}}(1)}(x_1,0,0,1,1)$ would induce the negation and be frugal.

Proof by example

Case 2: The negation is not induced by a frugal polynomial.

Case 2.2: $s^{\mathbf{A}}(1,0,1,1,1) = 1$:

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Proof by example

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Note that if $s^{\mathbf{A}}(0,0,1,1,1) = 1$, then $s^{\mathbf{T}_{\mathbf{A}}(1)}(x_1, 0, 1, 1, 1)$ would induce the negation and be frugal.

Applications to the study of the complexity of $\textsc{PolSat}\ \mathbf{A}$

The polynomial satifiability problem

The problem $\textsc{PolSat}(\mathbf{A})$ is the following search problem:

The polynomial satifiability problem

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The polynomial satifiability problem

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The complexity parameter of $\textsc{PolSat}(\mathbf{A})$ is $|p| + |q|$.

The Exponential Time Hypothesis

The exponential time hypothesis implies that there exists no sub-exponential time algorithm that solves $3\mathrm{SAT}$.

When is PolSat($\mathbf{A}$) not solvable in sub-exponential time

Theorem
[Gorazd, Krzaczkowski, 2011]

Let $\mathbf{A}$ be an algebra on $\{0, 1\}$.

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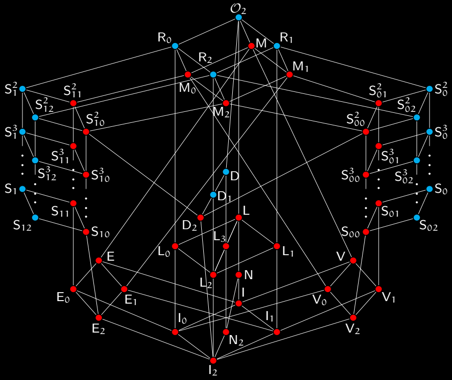
$\text{Clo}\,\mathbf{A} \Rightarrow \textsc{PolSat}(\mathbf{A})$ in P.



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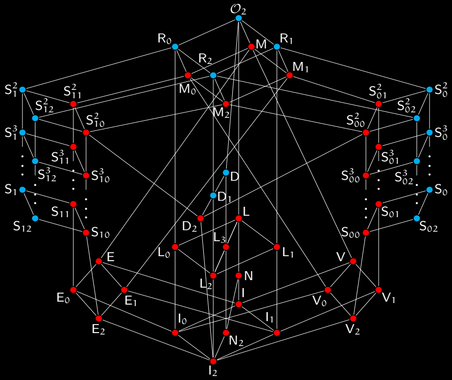

When is PolSat$(\mathbf{A})$ not solvable in sub-exponential time

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$\mathrm{Clo}\,\mathbf{A} \Rightarrow$
no sub-exponential time algorithm that solves $\textsc{PolSat}(\mathbf{A})$ under ETH.