Bounding the free spectrum of nilpotent algebras of prime power order

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Nilpotency

We will compare three properties of an algebra $\mathbf{A}$.

(1) $\mathbf{A}$ is **nilpotent**.

(2) $\mathbf{A}$ is **supernilpotent**.

(3) $\mathbf{A}$ is finite and has **small free spectrum**.

All three properties are **equivalent** for finite **groups** and **rings**.

Small free spectrum

Definition 1. A finite algebra $\mathbf{A}$ has **small free spectrum** if $\exists p \in \mathbb{R}[x] \, \forall n \in \mathbb{N} : |\mathrm{Clo}_n(\mathbf{A})| \leq 2^{p(n)}$.

(1) $\mathbf{A} = (\mathbb{Z}_2, +)$. Then $\mathrm{Clo}_n(\mathbf{A}) = 2^n$. Hence $\mathbf{A}$ has **small free spectrum.**

(2) $\mathbf{A} = (\mathbb{Z}_2, +, *, 0, 1)$. Then $\mathrm{Clo}_n(\mathbf{A}) = 2^{2^n}$. Hence $\mathbf{A}$ has **no small free spectrum.**

Supernilpotency of expanded groups

A function $f : A^n \to A$ is **absorbing** if for all $x_1, \ldots, x_n \in A$:

$$0 \in \{x_1, \ldots, x_n\} \Rightarrow f(x_1, \ldots, x_n) = 0.$$

Definition 2 (EA, Ecker, 2006)**.** An expanded group $\mathbf{A}$ is **supernilpotent** if there is a $k \in \mathbb{N}$ such that for all $n > k$, 0 is the only n-ary absorbing polynomial function of $\mathbf{A}$.

Example. $\mathbf{A} := (\mathbb{Z}_4, +, 2xy)$. Then every $f \in \mathrm{Pol}_n(\mathbf{A})$ can be written as

$$f(x_1, \ldots, x_n) := a + \sum_{i=1}^{n} b_i x_i + 2 \sum_{i \leq j} c_{i,j} \, x_i x_j.$$

"Hence" all 3-ary absorbing polynomial functions are 0 everywhere. Thus $\mathbf{A}$ is supernilpotent.

Supernilpotency of arbitrary algebras

Definition 3. $\mathbf{A}$ is 2-**supernilpotent** if for all vectors a_1, b_1, a_2, b_2, a_3, b_3 from A and for all term functions f of $\mathbf{A}$ we have

$$\left.\begin{array}{r} f(a_1,a_2,a_3) = f(a_1,a_2,b_3) \\ f(b_1,a_2,a_3) = f(b_1,a_2,b_3) \\ f(a_1,b_2,a_3) = f(a_1,b_2,b_3) \end{array}\right\} \Rightarrow f(b_1,b_2,a_3) = f(b_1,b_2,b_3).$$

This is an **infinite set** of **quasi-identities**, which are equivalent to the **higher commutator identity** $[\alpha,\beta,\gamma] \approx 0$.

k-**supernilpotent** is defined in the same way.
$\mathbf{A}$ **supernilpotent** $\Leftrightarrow$ $\exists k \in \mathbb{N}$: $\mathbf{A}$ is k-supernilpotent.

Nilpotency

Defined through the **binary commutator operation**.

Definition 4. For ideals A, B of an expanded group $\mathbf{V}$, the **commutator** $[A, B]$ is the ideal of $\mathbf{V}$ generated by

$$\{p(a, b) \mid a \in A,\, b \in B,\, p \in \mathsf{Pol}_2(\mathbf{V}), \forall v \in V : p(0, v) = p(v, 0) = 0\}.$$

For arbitrary algebras defined by the **commutator identity**

$$\exists k \in \mathbb{N} \,:\, [\ldots[[\alpha_1, \alpha_2], \alpha_3], \ldots, \alpha_{k+1}] \approx 0.$$

Supernilpotency and small free spectrum

Theorem 5. A finite algebra in a cm variety is supernilpotent iff it has small free spectrum.

Proofs:

For **expanded groups**, use a combinatorial argument by G. Higman that connects the number of **absorbing polynomial functions** to the number of **polynomial functions** [EA, 2014].

In **cp varieties**, combine results by [Berman Blok, 1987], [Freese McKenzie, 1987], [Hobby McKenzie, 1988], [EA, Mudrinski, 2010].

Supernilpotency and small free spectrum

Theorem 5. A finite algebra in a cm variety is supernilpotent iff it has small free spectrum.

Proofs:

In **cm varieties**, use [Wires, arXiv 2017] to obtain that a finite snp algebra in a cm variety has a Mal'cev term.

Detailled discussion in [EA, Mudrinski, Opršal, arXiv 2017] and [EA, arXiv 2018].

Supernilpotency implies Nilpotency (?)

Theorem 6 (EA, Mudrinski, 2010)**.** Every supernilpotent algebra in a cp variety is nilpotent.

Proof through the **nested commutators** property (HC8) $[\alpha_1, \dots, \alpha_{i-1}, [\alpha_i, \dots, \alpha_k]] \leq [\alpha_1, \dots, \alpha_k]$.

Theorem 7 (Wires, arXiv 2017)**.** Every supernilpotent algebra in a cm variety is nilpotent.

Proof: By Theorem 4.11 of [Wires, arXiv 2017], every supernilpotent algebra in a cm variety has a Mal'cev term.

Supernilpotency implies Nilpotency

Theorem 8 (Kearnes, 1999, in a formulation justified by [Wires, arXiv 2017])**.** Every finite supernilpotent algebra in a cm variety is a direct product of nilpotent algebras of prime power order.

On the implication Nilpotency $\Rightarrow$ Supernilpotency

The following algebras are nilpotent and not supernilpotent:

- $\mathbf{B} = (\mathbb{Z}_4, +, 2x_1x_2, 2x_1x_2x_3, \ldots)$ satisfies $[[1,1],1] = 0$ and is nilpotent.

  $2x_1x_2 \ldots x_n$ is absorbing but not zero.

On the implication Nilpotency $\Rightarrow$ Supernilpotency

The following algebras are nilpotent and not supernilpotent:

- $\mathbf{S} = (\mathbb{Z}_6, +, (-1)^x)$ has $\equiv_2$ as a central congruence, and is hence nilpotent.
  $f(x_1, \ldots, x_n) := \prod_{i=1}^{n}(1-(-1)^{x_i})$ is absorbing, non zero, and polynomial because of
  $$\prod_{i=1}^{n}(1-(-1)^{x_i}) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \cdot (-1)^{\sum_{i \in I} x_i} .$$

Comparison

In a cm variety, we have

- supernilpotent $\Leftrightarrow$ small free spectrum (for finite $\mathbf{A}$),
- supernilpotent $\Rightarrow$ nilpotent,
- supernilpotent and finite $\Rightarrow$ product of prime power order,
- nilpotent $\not\Rightarrow$ supernilpotent,
- nilpotent $\not\Rightarrow$ small free spectrum.

On the implication Nilpotency $\Rightarrow$ Supernilpotency

In a cm variety, we have

- nilpotent, finite type, finite, product of ppo algebras $\Rightarrow$ small free spectrum [Blok, Berman, 1987].

For the rest of the talk, we will discuss this result.

On the implication Nilpotency $\Rightarrow$ Supernilpotency

Theorem 9 (Berman Blok, 1987)**.** Let $\mathbf{A}$ be a finite nilpotent algebra of finite type in a congruence modular variety. If $\mathbf{A}$ is a direct product of algebras of prime power order, then $\mathbf{A}$ has small free spectrum.

Proof relies on:

- A generalization of Higman's combinatorial argument from **groups** to **congruence uniform algebras**.

- A bound on the length of **commutator terms** found by [Vaughan-Lee, 1983] and [Freese, McKenzie, 1987, Chapter 14]. Used in proving that such an $\mathbf{A}$ is **finitely based**.

On the implication Nilpotency $\Rightarrow$ Supernilpotency

Theorem 9 (Berman Blok, 1987)**.** Let $\mathbf{A}$ be a finite nilpotent algebra of finite type in a cm variety. If $\mathbf{A}$ is a direct product of algebras of prime power order, then $\mathbf{A}$ has small free spectrum.

Question: How small?

On the implication Nilpotency $\Rightarrow$ Supernilpotency

Theorem 10 (EA, arXiv 2018)**.**
Let $\mathbf{A}$ be in a cm variety, nilpotent, $|A| = q$ prime power, all fundamental operations of arity $\leq m$. Let

$$\begin{aligned} h &:= \text{height of Con}\,(\mathbf{A}), \text{ and} \\ s &:= (m(q-1))^{h-1}. \end{aligned}$$

Then

- $\mathbf{A}$ is s-supernilpotent.

- $\exists p \in \mathbb{R}[x] : \deg(p) \leq s$ and $\forall n \in \mathbb{N} : |\mathrm{Clo}_n(\mathbf{A})| = 2^{p(n)}$.

Nilpotent, finite type, ppo $\Rightarrow$ supernilpotent

Ingredients of the proof: **Coordinatization**

Theorem 11 (EA, arXiv 2018)**.**
$\mathbf{A} = (A; F)$ nilpotent algebra in a cp variety, p prime, $|A| = p^n$.

Then there is $+$ such that $\mathbf{A}' = (A; F \cup \{+\})$ is still nilpotent, and $(A; +) \cong (\mathbb{Z}_p{}^n; +)$.

Proof: 5 pages.

Nilpotent, finite type, ppo $\Rightarrow$ supernilpotent

Ingredients of the proof:

By the coordinatization result, we assume that

$$\mathbf{A} = (\mathsf{GF}(p^n); +, (f_i^{\mathbf{A}})_{i \in I})$$

with

$$f_i(x_1, \ldots, x_n) \in \mathsf{GF}(p^n)[x_1, \ldots, x_n].$$

We show that there is a bound on absorbing polynomials of $\mathbf{A}$.

Nilpotent, finite type, ppo $\Rightarrow$ supernilpotent

Key idea of the proof:

$\mathbf{A} = (\mathsf{GF}(p^n); +, (f_i)_{i \in I})$.

We find $(h_j)_{j \in J}$ such that $\mathbf{A}$ is term equivalent to

$\mathbf{A}' = (\mathsf{GF}(p^n); +, (h_j)_{j \in J})$ such that

- all h_j are absorbing,
- $\mathsf{Clo}_A(\{h_j | j \in J\})$ generates $\mathsf{Clo}(\mathbf{A})$ as an additive group,
- we have a bound on the arity of h_j.

Nilpotent, finite type, ppo $\Rightarrow$ supernilpotent

Ingredients of the proof:

$$\mathbf{A} = (\mathsf{GF}(p^n); +, (h_j)_{j \in J})$$

$$g(x_1, x_2, x_3) = h_1(h_2(x_1, x_2), x_3)$$

... bound on the arity because of nilpotency.

$$\begin{aligned}
&h(x_1 + x_2, x_3) \\
&= h(x_1 + x_2, x_3) - h(x_1, x_3) - h(x_2, x_3) + h(x_1, x_3) + h(x_2, x_3) \\
&= h'(x_1, x_2, x_3) + h(x_1, x_3) + h(x_2, x_3)
\end{aligned}$$

... every function is a sum of compositions of absorbing functions.

Nilpotent, finite type, ppo $\Rightarrow$ supernilpotent

Technical simplification:

Work with polynomials over GF(p^n) instead of functions: we then have monomials, degree, ... and

clones of polynomials, Couceiro-Foldes product of function sets, Associativity Lemma, decomposition of a polynomial into its **homovariate components**.

[9 pages]

On the implication Nilpotency $\Rightarrow$ Supernilpotency

Theorem 9 (Berman Blok, 1987)**.** Let $\mathbf{A}$ be a finite nilpotent algebra of finite type in a cm variety. If $\mathbf{A}$ is a direct product of algebras of prime power order, then $\mathbf{A}$ has small free spectrum.

Question: How small? **Answer:** $2^{p(n)}$ with

$$\mathrm{deg}(p) \leq (m(|A|-1))^{\log_2(|A|)-1},$$

where m is the maximal arity of fundamental operations of $\mathbf{A}$ (if $|A| > 1$).

[EA, Bounding the free spectrum of nilpotent algebras of prime power order, *arXiv*, 2018]