# Clonoids

Erhard Aichinger

Institute for Algebra
Johannes Kepler University Linz, Austria

Supported by the Austrian Science Fund FWF
P 29931

## The origin of the word "clonoids"

In 2009, EA & Peter Mayr & Ralph McKenzie proved:

- Every finite algebra with a Mal'cev term is finitely related.

- $$\left.\begin{array}{l} C \text{ clone on a finite set } A, \\ C \text{ contains an edge term} \end{array}\right\} \Rightarrow \exists n \in \mathbb{N},\ \rho \subseteq A^n : C = \mathsf{Pol}\,(\{\rho\}).$$

- Let $k \in \mathbb{N}$, $A$ finite set,

  $$\mathcal{C}_k := \{C \mid C \text{ clone on } A \text{ with } k\text{-edge term}\}.$$

  Then $(\mathcal{C}_k, \subseteq) \models$ (DCC).

## (DCC) yields finitely related

**Proposition 1.** Let $A$ be a finite set, $C$ clone on $A$. If

$$(\{D \mid D \text{ clone on } A, C \subseteq D\}, \subseteq )$$

has (DCC), then $C$ is finitely related.

*Proof:* Pol Inv$^{[1]}C \supseteq$ Pol Inv$^{[2]}(C) \supseteq$ Pol Inv$^{[3]}(C) \supseteq \cdots$ becomes constant after $N$ steps. Then $C =$ Pol Inv$^{[N]}C$.

## Inner working of the "Finite relatedness"-proof

Let $C$ be a clone on the finite set $A$.

**(1) Encode clones by forks:**

- Order $A$ linearly, and $A^n$ lexicographically.

- Forks$(C, \boldsymbol{x}) := \{(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) \mid f_1, f_2 \in C, \forall \boldsymbol{z} < \boldsymbol{x} : f_1(\boldsymbol{z}) = f_2(\boldsymbol{z})\}$.

**(2) Recover clone from forks:**

**Lemma 2** (Fork lemma)**. Suppose:** $C$ has Mal'cev operation, $C \subseteq D$, $\forall \boldsymbol{x} \in A^*$ : Forks$(C, \boldsymbol{x})$ = Forks$(D, \boldsymbol{x})$.
**Then** $C = D$.

## Inner working of the “Finite relatedness”-proof

- There is a connection between forks of different arity.

- **Use:** if $f \in C^{[19]}$, then

  $$\begin{aligned}&((x_1, x_2, x_3, x_4, x_5, x_6) \mapsto\\ &\quad f(\boldsymbol{x}_1, x_1, \boldsymbol{x}_2, x_1, \boldsymbol{x}_3, x_1, x_3, \boldsymbol{x}_4, x_4, x_3, x_1, x_4, \boldsymbol{x}_5, x_1, x_4, x_4, \boldsymbol{x}_6, x_4, x_4))\\ &\quad\; \in C^{[6]}.\end{aligned}$$

- This implies $\mathrm{Forks}(C, \boldsymbol{aabcca}) \supseteq \mathrm{Forks}(C, \underline{\boldsymbol{a}}a\underline{\boldsymbol{a}}a\underline{\boldsymbol{b}}ab\underline{\boldsymbol{c}}cbac\underline{\boldsymbol{c}}acc\underline{\boldsymbol{a}}cc)$ for all $a, b, c \in A$. (= **Embedded forks lemma**).

## Analysis of the proof

Where do we use:

- Mal'cev/edge term? *Allows translation from* **forks** *to* **clones** *via Fork Lemma*.
- clones are closed under $\nabla$ (cylindrification), $\Delta$ (identification), $\zeta$, $\tau$ (permutation)? *Allows to use Embedded Forks Lemma*.
- clones are closed under $*$ (composition)? *NOT USED AT ALL!*

# Finitely generated varieties

**Theorem 3** (EA, Peter Mayr, 2014)**.** Let $\mathbf{A}$ be a finite algebra with edge term, $W \leq V(\mathbf{A})$. Then $\exists \mathbf{B}$: $W = V(\mathbf{B})$ and $\mathbf{B}$ finite.

*Idea of the proof:* $W$ is determined by its equational theory Th($W$).

## The equational theory as a set of functions

$$\mathsf{Th}(W) = \{(s,t) \mid n \in \mathbb{N}, s, t \ n\text{-ary terms}, W \models s \approx t\}.$$

Inside $\mathbf{A}$:

$\mathsf{Th}_{\mathbf{A}}(W) = \{\boldsymbol{a} \mapsto \begin{pmatrix} s^{\mathbf{A}}(\boldsymbol{a}) \\ t^{\mathbf{A}}(\boldsymbol{a}) \end{pmatrix} \mid n \in \mathbb{N},\ s, t\ n\text{-ary terms},\ W \models s \approx t\}.$

$$\mathsf{Th}_{\mathbf{A}}(V(\mathbf{A})) = \{(c,c) \mid c \in \mathsf{Clo}(\mathbf{A})\}$$

$$\mathsf{Th}_{\mathbf{A}}(\mathsf{Mod}(\mathsf{x} \approx \mathsf{y})) = \{(c,d) \mid n \in \mathbb{N}, c, d \in \mathsf{Clo}_n(\mathbf{A})\}.$$

## Finitely generated varieties

$\Phi : W \mapsto \mathrm{Th}_{\mathbf{A}}(W)$ is **injective** and **order reverting**. What is the image of $\Phi$?

**Definition 4** (Clonoids; EA, Mayr, JPAA 2016)**.** Let $\mathbb{B}$ be an algebra, $A \neq \varnothing$. For a subset $C$ of $\bigcup_{n\in\mathbb{N}} B^{A^n}$ and $k \in \mathbb{N}$, we let $C^{[k]} := C \cap B^{A^k}$. We call $C$ a **clonoid with source set $A$ and target algebra $\mathbb{B}$** if

(1) for all $k \in \mathbb{N}$: $C^{[k]} \leq \mathbf{B}^{A^k}$,

(2) for all $k, n \in \mathbb{N}$, $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$, $c \in C^{[k]}$, $((a_1, \ldots, a_n) \mapsto c(a_{i_1}, \ldots, a_{i_k})) \in C^{[n]}$.

## Closed sets of finitary functions

**Definition 5** (**Composition**, Miguel Couceiro and Stephan Foldes, Acta Cybernetica 2007)**.**

$\mathsf{Fin}(A, B) := \bigcup_{n \in \mathbb{N}} B^{A^n}$

$X \subseteq \mathsf{Fin}(A, B)$, $Y \subseteq \mathsf{Fin}(B, C)$

$YX := \{g(f_1, \ldots, f_n) \mid m, n \in \mathbb{N}, g \in Y^{[n]}, f_1, \ldots, f_n \in X^{[m]}\}$.

**Lemma 6** (**Associativity Lemma**, Couceiro & Foldes)**.**
Let $J$ be the projection clone.
Then $(XY)Z \subseteq X(YZ) \subseteq (X(YJ))Z$.

## Names of closed sets of finitary functions

**Definition 7.** $X \subseteq \mathsf{Fin}(A, A)$ is a

- **clone** $\Leftrightarrow J \subseteq X$, $XX \subseteq X$.
- **iterative algebra** (Harnau) $\Leftrightarrow X(J \cup X) \subseteq X$ and $X \neq \varnothing$.

Other flowers in the garden $\mathcal{P}(\mathsf{Fin}(A, A))$:

- **rose** $\Leftrightarrow XJ \subseteq X$, $XX \subseteq X$,
- **daisy** $\Leftrightarrow XX \subseteq X$,
- **iris** $\Leftrightarrow XX \subseteq X$ and closed under $\tau, \zeta, \nabla$.

On the group $S_3$, $X := \{\boldsymbol{x} \mapsto [x_i, x_j] \mid i, j \in \mathbb{N}\}$ is a rose, but not an iterative algebra, since $[[x_1, x_2], x_3] = [x_1, x_2] * [x_1, x_2]$ is in $X(X \cup J)$, but not in $XX$.

## Definition of clonoids

**Definition 8.** Let $A \neq \varnothing$, $\mathbf{B}$ an algebra. $C \subseteq \mathsf{Fin}(A, B)$ is a **clonoid** $\Leftrightarrow$ $\mathsf{Clo}(\mathbf{B})\, C \subseteq C$, $C J_A \subseteq C$.

Relational description of clonoids: (Couceiro, Foldes, Discuss. Math. Gen. Algebra Appl. 29 (2009)).

- **dual objects:** relational pairs $(R, S)$ with $S \subseteq A$, $S \leq \mathbf{B}^n$,

- **closure of dual objects:** $(J_A, \mathsf{Clo}(\mathbf{B}))$-conjunctive minors.

## Finiteness result for clonoids

**Theorem 9** (EA, Peter Mayr, JPAA 2016). Let $A$ be a finite set, $\mathbf{B}$ a finite algebra with edge term. Let $\mathcal{C}$ be the set of clonoids with source $A$ and target algebra $\mathbf{B}$. Then $(\mathcal{C}, \subseteq)$ has the (DCC).

Corollaries:

- Finite algebras with edge term are finitely related.
- Subvarieties of f.g. varieties with edge term are f.g.

## Further consequences of "DCC for subclonoids"

- Every iterative algebra on a finite algebra that is closed under an edge term is determined by one finite relation pair $(R, S)$.

- $\cdots$

## Clonoids in the description of clones

**Goal:** Describe all clones on $(\mathbb{Z}_p \times \mathbb{Z}_q)$ that contain $+$.

**Results:**

- In the case $p = q$ all *constantive* clones have been determined by Andrei Bulatov (there are infinitely many).

- In the case $p \neq q$ all *constantive* clones have been determined by Peter Mayr & EA (there are 17).

## Clonoids inside clones

Let $C$ be a clone on $\mathbb{Z}_q \times \mathbb{Z}_p$.
$D := \{f : \mathbb{Z}_q^* \to \mathbb{Z}_p \mid (\boldsymbol{x}, \boldsymbol{y}) \mapsto (0, f(\boldsymbol{y})) \in C\}$.
$L_q :=$ Clo$(\mathbb{Z}_q, +)$, $L_p :=$ Clo$(\mathbb{Z}_p, +)$.

Then $L_p(DL_q) \subseteq D$.

Hence $D$ is a clonoid from $\mathbb{Z}_p$ to $(\mathbb{Z}_q, +)$ because $L_p(DJ) \subseteq D$.

**Definition 10.** $D$ is a $(p, q)$-**linearly closed clonoid** from $\mathbb{Z}_p$ to $(\mathbb{Z}_q, +) :\Longleftrightarrow L_p(DL_q) \subseteq D$.

## Present + Future

The description of $(p, q)$-closed linearly closed clonoids is useful for:

- Describe all clones on $\mathbb{Z}_p$ with $+$.
- Describe all clones on $\mathbb{Z}_p \times \mathbb{Z}_q$ with $+$.
- Provide examples of a finite set with infinitely many nonfinitely generated Mal'cev clones.

Parts of this program have been realized by Sebastian Kreinecker and Stefano Fioravanti.