The Complexity of Solving Equations over Finite Groups

A collecion of results by Goldmann and Russell from 1999

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Introduction

Problem

We will assume that $(G, \cdot)$ is a finite group.

Definition (Horváth and Szabó 2006)

Given polynomials $p_1, \ldots, p_r, q_1, \ldots, q_r$ over G we want to decide if there is an $x = (x_1, \ldots, x_n) \in G^n$ such that

$$p_i(x) = q_i(x), \quad \text{for all } i = 1, \ldots, r.$$

We write $\textsc{PolSysSat}(G)$ for short. If $r = 1$ we write $\textsc{PolSat}(G)$.

What is a polynomial over G? Each polynomial p over G is of the form

$$p = w_1 \cdot w_2 \cdots w_s \text{ where } w_j \in G \cup \{x_1, \ldots, x_n\} \cup \{x_1^{-1}, \ldots, x_n^{-1}\}.$$

Hence we can assume that $q_i(x) = 1$, i.e. our system is given as

$$p_i(x) = 1, \quad \text{for all } i = 1, \ldots, r.$$

We ask: For which groups G is $\textsc{PolSysSat}(G) \in P$ and for which $\textsc{PolSat}(G) \in P$?

Examples

Some examples of polynomial equations include:

- $(\mathbb{Z}_8, +)$:
$$2 + 3x_1 + 5x_2 + 7x_3 = 0,$$
- $(D_4, \cdot)$ with $a^4 = b^2 = 1$ (so $|D_4| = 8$):
$$a \cdot a \cdot x_1 \cdot x_1 \cdot b \cdot x_2^{-1} \cdot b \cdot a = x_3^{-1} \cdot b,$$
- $(S_3, \circ)$:
$$x \circ \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \circ x^{-1} \circ \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},$$
- $(S_5, \circ)$:
$$x_1 \circ \begin{pmatrix} 1 & 5 \end{pmatrix} \circ x_2 \circ \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \circ \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

Goal of today

Goldmann and Russell proved two important theorems:

Theorem 1 (Goldmann and Russell 1999, Thm. 1+2)

If G is an abelian group, then $\textsc{PolSysSat}(G) \in P$ and $\textsc{PolSysSat}(G) \in NPC$ otherwise.

Theorem 2 (Goldmann and Russell 1999, Thm. 10 + Cor. 12)

If G is a nilpotent group, then $\textsc{PolSat}(G) \in P$ and if G is not solvable then $\textsc{PolSat}(G) \in NPC$.

System of Equations

Solving systems over abelian groups

As a first step we will show:

Theorem 1 (part 1, (Goldmann and Russell 1999, Thm. 1))

If G is an abelian group, then $\textsc{PolSysSat}(G) \in P$.

Proof: Every finite abelian group G can be written as

$$G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_l}.$$

Want to solve system $p_i(x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, r$ with polynomials p_i over G. Instead of solving the system over G we can rewrite it as l individual systems over $\mathbb{Z}_{n_k}$. Hence we only consider the case $\mathbb{Z}_m$. Over $\mathbb{Z}_m$ we can solve a system using (essentially) Gaussian elimination.

Solving systems over $\mathbb{Z}_m$

For a polynomial $\tilde{p}_i$ over $\mathbb{Z}_m$ we can write:

$$\tilde{p}_i(x_1, \dots, x_n) = p_i^{(1)} x_1 + \dots + p_i^{(n)} x_n - p_i^{(0)}.$$

Hence the system $\tilde{p}_i(x_1, \dots, x_n) = 0$ is equivalent to

$$(a_{ij})_{i,j=1}^{r,n} x := Ax := \begin{pmatrix} p_1^{(1)} & \dots & p_1^{(n)} \\ \vdots & & \vdots \\ p_r^{(1)} & \dots & p_r^{(n)} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} p_1^{(0)} \\ \vdots \\ p_r^{(0)} \end{pmatrix} =: b =: (b_i)_{i=1}^{r}.$$

We do not change the satisfiablity of the system if we:

- Interchange rows of A: Reordering equations.
- Interchange columns of A: Reordering variables.
- Adding multiple of row to different row.
- Adding multiple of column to different column.

Algorithm

For computing a diagonal form of the matrix using these operations do:

1. Find a nonzero minimal entry a_{ij} of A.
2. Reduce all entries in row i and column j.
3. If all entries in row i and column j (except a_{ij}) are zero, then swap row i with row 1 and column j with column 1 and proceed with step 1 with the submatrix arising by removing the first row and first column.
4. Otherwise we have created an element which is smaller than a_{ij}. Again proceed with step 1 with the whole matrix.

The elements in the matrix get strictly smaller, so the algorithm terminates. It has polynomial complexity $\mathcal{O}(rn \min(r, n))$.

Hence in total $\textsc{PolSysSat}(\mathbb{Z}_n) \in P$, so $\textsc{PolSysSat}(G) \in P$ for abelian groups G. $\square$

NP-completeness

The more difficult part of Theorem 1 will be:

Theorem 1 (part 2)

If G is an not abelian, then $\textsc{PolSysSat}(G)$ is NP complete.

How can one show NP-completeness? (Polynomially) reduce a problem which is known to be NP-complete to the problem for which we want to show NP-completeness. Here: Graph-Colorability.

Graph-colorability

Theorem (Karp 1972)

Given a graph G and $k \geq 3$ different colors. The problem of deciding if there is a color for each vertex of G such that two vertices which are connected by an edge do not have the same color is NP-complete.

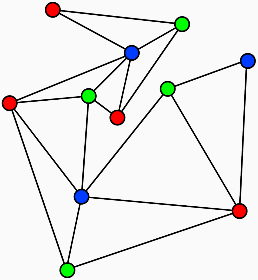


Figure 1: Source: Wikimedia Commons (David Eppstein), *https://commons.wikimedia.org/wiki/File:Triangulation_3-coloring.svg*

Small groups

order	abelian groups	non-abelian groups
1	$\mathbb{Z}_1$	
2	$\mathbb{Z}_2$	
3	$\mathbb{Z}_3$	
4	$\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$	
5	$\mathbb{Z}_5$	
6	$\mathbb{Z}_6$	$D_3 \cong S_3$
7	$\mathbb{Z}_7$	
8	$\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	D_4, Q_8
9	$\mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3$	
10	$\mathbb{Z}_{10}$	D_5
11	$\mathbb{Z}_{11}$	
12	$\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6$	D_6, A_4, T
13	$\mathbb{Z}_{13}$	
14	$\mathbb{Z}_{14}$	D_7
15	$\mathbb{Z}_{15}$	

Table 1: Hungerford 2003

PolSysSat(S_3)

To prove that PolSysSat(G) is *NP*-complete for non-abelian groups G we use induction on order of the groups. Smallest non-abelian group is S_3.

Lemma (Goldmann and Russell 1999, Thm. 3)

PolSysSat(S_3) is *NP*-complete.

Proof: We will show that coloring a graph with 6 colors can be reduced to PolSysSat(S_3). Every element in S_3 corresponds to a color (6 colors total). With each vertex i in the graph we associate a variable x_i. For each edge (i, j) in the graph we introduce two variables y_{ij}, z_{ij} and the equation

$$y_{ij}\, x_i\, x_j^{-1}\, z_{ij}\, x_j\, x_i^{-1}\, z_{ij}^{-1}\, y_{ij}^{-1} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}.$$

PolSysSat(S_3)

If the coloring is legal, then for every edge (i, j) we have $\alpha := x_i x_j^{-1} \neq \begin{pmatrix}1\end{pmatrix}$. The equation

$$y_{ij}\, \alpha z_{ij}\, \alpha^{-1}\, z_{ij}^{-1}\, y_{ij}^{-1} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}$$

has a solution if and only if α is not the identity:

α	z_{ij}	y_{ij}	α	z_{ij}	y_{ij}
$\begin{pmatrix}1 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}2 & 3\end{pmatrix}$	$\begin{pmatrix}2 & 3\end{pmatrix}$
$\begin{pmatrix}1 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1 & 3 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 3\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
$\begin{pmatrix}2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$			

Hence we have reduced the problem of coloring a graph to the problem of solving a system of equations over S_3. If we can solve the system of equations over S_3 we can color the graph. Therefore PolSysSat(S_3) $\in NPC$. $\square$

Inducible subgroups

Having the base-case S_3 settled we will introduce some more concepts before we will prove the general result.

Definition (Goldmann and Russell 1999, Def. 1)

A subset $H \subseteq G$ is called inducible if there is a polynomial p over G such that

$$H = \mathrm{Im}(p) = \{p(g_1, \ldots, g_n) \;:\; g_1, \ldots, g_n \in G\}.$$

Inducible subgroups have the nice property that NP completeness carries over to the larger group. Namely:

Lemma (Goldmann and Russell 1999, Lemma 4)

Let H be an inducible subgroup of G.

1. If $\textsc{PolSysSat}(H) \in NPC$, then $\textsc{PolSysSat}(G) \in NPC$.
2. If H is a normal subgroup of G and $\textsc{PolSysSat}(G/H) \in NPC$, then $\textsc{PolSysSat}(G) \in NPC$.

Proof complexity of inducible subgroups

Proof of $\textsc{PolSysSat}(H) \in NPC \implies \textsc{PolSysSat}(G) \in NPC$:

Since H is inducible there exists a polynomial $p(x_1, \ldots, x_n)$ over G such that $H = \mathrm{Im}(p)$. Given an equation

$$w_1 \cdot w_2 \cdots w_s = 1 \text{ over } H \text{ with } w_i \in H \cup \{y_1, \ldots, y_m\} \cup \{y_1^{-1}, \ldots, y_m^{-1}\}$$

we can replace every occurrence of y_i with $p(x_1^{(i)}, \ldots, x_n^{(i)})$ where $x_j^{(i)}$ are new variables over G and every occurrence of y_i^{-1} with $p(x_1^{(i)}, \ldots, x_n^{(i)})^{-1}$. Then we have a new equation over G which can be satisfied if and only if the original one can be satisfied. $\square$

Proof complexity of inducible subgroups

Proof of $\textsc{PolSysSat}(G/H) \in NPC \implies \textsc{PolSysSat}(G) \in NPC$:

Now an equation over G/H looks like

$$(w_1 \cdot w_2 \cdots w_s)H = w_1 H \cdot w_2 H \cdots w_s H = H$$

with $w_i \in G \cup \{y_1, \ldots, y_m\} \cup \{y_1^{-1}, \ldots, y_m^{-1}\}$ which we can rewrite as

$$w_1 \cdot w_2 \cdots w_s = p(x_1, \ldots, x_n)$$

and

$$w_1 \cdot w_2 \cdots w_s \cdot p(x_1, \ldots, x_n)^{-1} = 1$$

over G for new variables $x_1, \ldots, x_n$. $\square$

Commutators

Definition

For two elements $a, b \in G$ we write

$$[a, b] := aba^{-1}b^{-1}$$

and call $[a, b]$ a commutator.

For two subsets $A, B \subseteq G$ we write

$$[A, B] := \{[a, b] = aba^{-1}b^{-1} \ : \ a \in A, b \in B\}$$

and $(A, B) = \langle [A, B] \rangle$ for the group generated by the commutators $[a, b]$ and call (A, B) a commutator subgroup.

In particular (G, G) is the commutator subgroup of G. In fact (G, G) is the smallest subgroup of G such that $G/(G, G)$ is abelian. Furthermore $(G, G) = \{1\}$ if and only if G is abelian.

Commutator subgroup

Lemma (Goldmann and Russell 1999, Lemma 5)

$(G, G) \subseteq G$ is inducible.

Reminder: $[a, b] = aba^{-1}b^{-1}$ and $(G, G) = \langle\{[a, b] \,:\, a, b \in G\}\rangle$.

Proof: Every element $g \in (G, G)$ can be written as

$$g = [a_1, b_1][a_2, b_2] \cdots [a_m, b_m].$$

Since G is finite and $[a, a] = 1$ we have a fixed $m \in \mathbb{N}$ such that

$$(G, G) = \{[a_1, b_1][a_2, b_2] \cdots [a_m, b_m] \,:\, a_i, b_i \in G\}.$$

Hence we can choose the polynomial

$$p(x_1, y_1, \ldots, x_m, y_m) := [x_1, y_1][x_2, y_2] \cdots [x_m, y_m].$$

This p induces (G, G), i.e. $p(G^{2m}) = (G, G)$. $\square$

Commutator facts

Later we will need the commutator subgroups

$$(a, G) := (\{a\}, G) = \{[a, g_1][a, g_2] \cdots [a, g_m] \;:\; g_i \in G\}.$$

Lemma (Goldmann and Russell 1999, Lemma 6)

Let $a \in G$. Then

1. $(a, G) \subseteq (G, G)$,
2. (a, G) is inducible and
3. (a, G) is normal in G.

Proof of $(a, G) \subseteq (G, G)$: For

$$[a, g_1][a, g_2] \cdots [a, g_m] \in (a, G)$$

we also have

$$[a, g_1][a, g_2] \cdots [a, g_m] \in (G, G).$$

Commutator facts

Proof of (a, G) *inducible*: We can choose the polynomial

$$p(x_1, \ldots, x_m) := [a, x_1][a, x_2] \cdots [a, x_m],$$

then $p(G^m) = (a, G)$.

Proof of (a, G) *normal:* Since $(a, G) = \langle \{[a, g] \ : \ g \in G\} \rangle$, it is sufficient to show $b[a, g]b^{-1} \in (a, G)$ for all $g, b \in G$. This follows as

$$\begin{aligned} b[a, g]b^{-1} = b(aga^{-1}g^{-1})b^{-1} &= (ba\,\underbrace{b^{-1}a^{-1})(ab}_{=1}\,ga^{-1}g^{-1}b^{-1}) \\ &= (aba^{-1}b^{-1})^{-1}(abga^{-1}g^{-1}b^{-1}) = [a, b]^{-1}[a, bg]. \end{aligned}$$

As (a, G) is a subgroup $[a, b]^{-1} \in (a, G)$, so $b[a, g]b^{-1} \in (a, G)$ and (a, G) is normal in G. □

Commutator simple

Definition

We call

$$Z(G) := \{g \in G \,:\, gh = hg \text{ for all } h \in G\}$$

the center of G.

Definition (Goldmann and Russell 1999, Def. 3)

We call G commutator simple if for all $a \notin Z(G)$ we have $(G, G) = (a, G)$.

The last Lemma we need before we can finish the proof that $\textsc{PolSysSat}(G) \in NPC$ for non-abelian G:

Lemma (Goldmann and Russell 1999, Lemma 7)

Let G be a non-abelian commutator simple group. Then $\textsc{PolSysSat}(G) \in NPC$.

Commutator simple

Proof: If G is non-abelian, then $G/Z(G)$ is not cyclic. Therefore $G/Z(G)$ contains at least four elements, we will write $k = |G/Z(G)|$. Again we reduce the colorability of a graph with k colors to solving systems over $G/Z(G)$. For every vertex v in the graph we introduce a variable x_v. Then $x_v Z(G) \in G/Z(G)$ will determine the color of v. So two vertices v, w will have the same color if and only if $x_v x_w^{-1} \in Z(G)$.

If $x_v x_w^{-1} \notin Z(G)$, then $(x_v x_w^{-1}, G) = (G, G)$ as G is commutator simple.

Otherwise if $x_v x_w^{-1} \in Z(G)$, then for all $g \in G$ we have

$$[x_v x_w^{-1}, g] = x_v x_w^{-1} g x_w x_v^{-1} g^{-1} = g x_v x_w^{-1} x_w x_v^{-1} g^{-1} = 1,$$

so $(x_v x_w^{-1}, G) = \{1\}$.

Commutator simple

There is a constant $m \in \mathbb{N}$ such that

$$(a, G) = \{[a, g_1][a, g_2] \cdots [a, g_m] \ : \ g_i \in G\}.$$

Let $1 \neq b \in (G, G)$. This b exists as G is not abelian. Than for every edge $e = (v, w)$ in the graph we introduce the equation

$$[x_v x_w^{-1}, s_1^e] \cdots [x_v x_w^{-1}, s_m^e] = b$$

over G where the s_i^e are new variables.

If this system has a solution, then $x_v x_w^{-1} \notin Z(G)$, because if $x_v x_w^{-1} \in Z(G)$, then $b \notin (x_v x_w^{-1}, G) = \{1\}$. So in this case we have legal coloring with $k \geq 4$ colors.

On the other hand, if it has a legal coloring, i.e. $x_v x_w^{-1} \notin Z(G)$, then we can find a solution of the system since in this case $(x_v x_w^{-1}, G) = (G, G)$.

So we have reduced the colorability problem of a graph to the problem of solving a system of equations over G, so $\textsc{PolSysSat}(G) \in NPC$. $\square$

Solving systems over non-abelian groups

Theorem 1 (part 2, Goldmann and Russell 1999, Thm. 2)

If G is an not abelian, then PolSysSat(G) is NP complete.

Proof: By Induction over the group order. For the smallest non-abelian group S_3 we have already shown it.

So assume that the theorem holds for all non-abelian groups of order $n-1$ or less and let G be a non-abelian group of order n. If G is commutator simple, the previous lemma has shown that PolSysSat$(G) \in NPC$.

So we assume that G is not commutator simple. Hence there exists $a \in G - Z(G)$ with $(a, G) \subsetneq (G, G)$. Then (a, G) is nontrivial, because if $[a, g] = 1$ for every $g \in G$, then $a \in Z(G)$, a contradiction.

Then $G/(a, G)$ is non-abelian as $(a, G) \subsetneq (G, G)$. As $|G/(a, G)| < n$ we have PolSysSat$(G/(a, G)) \in NPC$ by induction. Since (a, G) is a normal inducible subgroup of G by a previous Lemma we have PolSysSat$(G) \in NPC$. $\square$

Single Equation

Single equation

Theorem 2

If G is a nilpotent group, then $\textsc{PolSat}(G) \in P$ and if G is not solvable then $\textsc{PolSat}(G) \in NPC$.

Before we can look at the proof we need to understand what nilpotent and solvable groups are.

Nilpotent groups

Definition

Let $G_0 := G$ and

$$G_{i+1} := (G, G_i) = \langle \{[g,h] = ghg^{-1}h^{-1} \;:\; g \in G, h \in G_i\} \rangle$$

for $i \geq 0$. Then G is called nilpotent if $G_n = \{1\}$ for some $n \in \mathbb{N}$.

The groups G_i form the lower central series.

Abelian groups are nilpotent as $G_1 = (G, G) = \{1\}$.

Let $p \in \mathbb{P}$ be a prime. A group of order p^n is nilpotent (and called a p-group). Since $|D_4| = 8 = 2^3$, the group D_4 is nilpotent. However, it is not abelian!

Solvable groups

We have already seen

$$(G, G) = \langle \{[g, h] = ghg^{-1}h^{-1} \ : \ g, h \in G\} \rangle.$$

Definition

Let $G^{(1)} := (G, G)$. By Induction we define

$$G^{(i+1)} := (G^{(i)}, G^{(i)}) := \langle \{[g, h] \ : \ g \in G^{(i)}, h \in G^{(i)}\} \rangle$$

and call $G^{(i)}$ the derived subgroups of G.

If $G^{(n)} = \{1\}$ for some $n \in \mathbb{N}$, then we call G solvable.

Abelian groups are solvable as $G^{(1)} = \{1\}$. Nilpotent groups are solvable.

Groups of order $p^n q^m$ for primes $p, q \in \mathbb{P}$ are solvable (Burnside). Groups of odd order are solvable (Feit-Thompson).

S_3 and S_4 are solvable but not nilpotent. S_n for $n \geq 5$ are not solvable.

Solvable groups

The derived subgroups $G^{(i)}$ form the derived series of G:

$$G \geq G^{(1)} \geq \cdots \geq G^{(n)} \geq \cdots .$$

If G is solvable, there is an $n \in \mathbb{N}$ such that $G^{(n)} = \{1\}$.

If G is not solvable there is (since G is finite) an $n \in \mathbb{N}$ such that

$$G^{(*)} := G^{(n)} = G^{(n+1)} = G^{(n+2)} = \cdots ,$$

i.e. $(G^{(*)}, G^{(*)}) = G^{(*)}$. By a previous Lemma applied inductively $G^{(*)}$ is an inducible subgroup of G.

Lemma

Let H be a normal subgroup of G. Then G is solvable if and only if H and G/H are solvable.

Nilpotent groups solvable

Lemma

A nilpotent group G is solvable.

Proof: We will show first by induction that $G^{(i)} \subseteq G_i$ for all i, i.e. derived series is under the lower central series.

Clearly $G^{(1)} = (G, G) = G_1$ by their definitions.

Now let $G^{(i)} \subseteq G_i$. Then

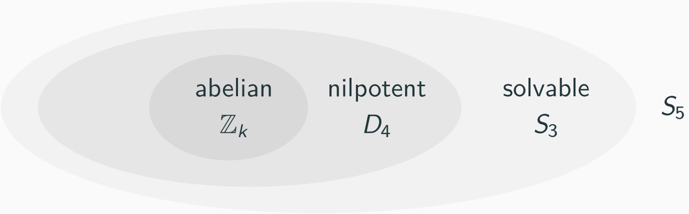
$$G^{(i+1)} = (G^{(i)}, G^{(i)}) \subseteq (G, G^{(i)}) \subseteq (G, G_i) = G_{i+1}.$$

Now if G is nilpotent, then $G_n = \{1\}$ for some $n \in \mathbb{N}$. Then $G^{(n)} \subseteq G_n = \{1\}$, so G is solvable. $\square$

Single equation

Theorem 2

If G is a nilpotent group, then $\textsc{PolSat}(G) \in P$ and if G is not solvable then $\textsc{PolSat}(G) \in NPC$.



Single equation *NPC* for non-solvable groups

Theorem 2, part 1

If G is not solvable then $\textsc{PolSat}(G) \in NPC$.

Again need some preparation.

Lemma (Goldmann and Russell 1999, Lemma 8)

Let H be an inducible subgroup of G.

1. If $\textsc{PolSat}(H) \in NPC$, then $\textsc{PolSat}(G) \in NPC$.
2. If H is normal in G and $\textsc{PolSat}(G/H) \in NPC$, then $\textsc{PolSat}(G) \in NPC$.

Proof : In the same way as for $\textsc{PolSysSat}$. $\square$

Commutator simple non-solvable groups

Reminder: G is commutator simple if $\forall a \notin Z(G)$: $(G,G) = (a,G)$.

Lemma (Goldmann and Russell 1999, Lemma 9)

Let G be a non-solvable group with $G = (G,G)$ and G is commutator simple. Then $\textsc{PolSat}(G) \in NPC$.

Proof : Similar to previous Lemma. $\square$

Single equation

Theorem 2 (part 1, Goldmann and Russell 1999, Thm. 10)

If G is not solvable then $\textsc{PolSat}(G) \in NPC$.

Proof: Again by induction on group order.

Basis: Let G be the smallest non-solvable group (which is A_5 with order 60). Then G must be simple, because otherwise there is a nontrivial normal subgroup H and then G/H as well as H would be solvable as G is chosen with minimal order. Since (G, G) is a normal subgroup and by assumption $(G, G) \neq \{1\}$ we must have $(G, G) = G$. As (a, G) are normal subgroups in G again we have $(a, G) = G$ for $a \notin Z(G)$:

Suppose $(a, G) = \{1\}$, then $[a, g] = aga^{-1}g^{-1} = 1$ for all $g \in G$, so $a \in Z(G)$, a contradiction.

Therefore by the previous Lemma $\textsc{PolSat}(G) \in NPC$ for $G = A_5$.

Single equation

Induction step: Consider arbitrary non-solvable group G. We look at $G^{(*)}$:

If $G^{(*)} \subsetneq G$, then by induction $\textsc{PolSat}(G^{(*)}) \in NPC$. Furthermore $G^{(*)}$ is an inducible subgroup of G, so $\textsc{PolSat}(G) \in NPC$.

If $G^{(*)} = G = (G, G)$ and G is commutator simple, the previous lemma showed $\textsc{PolSat}(G) \in NPC$. So we assume that G is not commutator simple, i.e. there is an $a \in G - Z(G)$ such that $(a, G) \subsetneq (G, G)$. As $a \notin Z(G)$ we have $(a, G) \neq \{1\}$, so $|G/(a, G)| < |G|$. As G is non-solvable either (a, G) or $G/(a, G)$ have to be non-solvable. By the induction hypothesis $\textsc{PolSat}((a, G)) \in NPC$ or $\textsc{PolSat}(G/(a, G)) \in NPC$. Again by a previous lemma $\textsc{PolSat}(G) \in NPC$. $\square$

Theorem 2 (part 2, Goldmann and Russell 1999, Cor. 12)

If G is nilpotent then $\textsc{PolSat}(G) \in P$.

Proof: See Goldman and Russell. $\square$

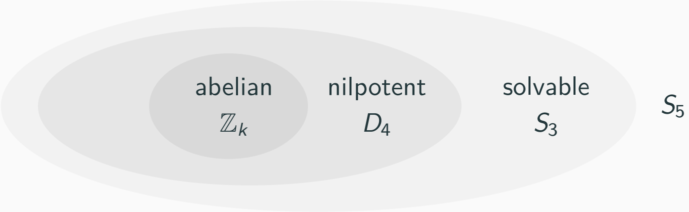
What happened in the last 20 years?

Nilpotent solvable groups

What about the nilpotent non-solvable groups? Goldmann and Russell did not know.

Still, we do not know. Ongoing research.

However, we already have examples of nilpotent non-solvable groups G for which $\textsc{PolSat}(G) \in P$, e.g. groups of order pq for primes p, q (Horváth and Szabó 2006). This shows that $\textsc{PolSat}(S_3) \in P$.



Generalising the results

Goldmann and Russell ”only” considered groups. What about other or more general algebras?

Example Rings

Let R be a finite ring.

- If R is nilpotent (i.e. $R^n = \{0\}$ for some $n \in \mathbb{N}$), then $\textsc{PolSat}(R) \in P$, otherwise $\textsc{PolSat}(R) \in NPC$ (Horváth 2011).
- If R is essentially an abelian group (i.e. $xy = 0$ for all $x, y \in R$), then $\textsc{PolSysSat}(R) \in P$ and $\textsc{PolSysSat}(R) \in NPC$ otherwise (Larose and Zádori 2006).

More general results can be found e.g. in Larose and Zádori 2006, Gorazd and Krzaczkowski 2011, Idziak and Krzaczkowski 2018, Aichinger 2019.

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