Near-rings of polynomial functions on expanded groups

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Near-ring modules

Definition

Let $\langle N, +, \cdot \rangle$ be a 0-symmetric near-ring with 1.
$\langle A, +, -, 0 \rangle$ is a compatible *N-group* if $\forall m, n \in N \ \forall a \in A$:

1. $m * a + n * a = (m + n) * a$
2. $(mn) * a = m * (n * a)$
3. $1 * a = a$
4. $\exists o \in N \ \forall b \in A : n * (a + b) - n * a = o * b$

Theorem (cf. Jacobson, Betsch, ..)

Assume A is finite simple compatible faithful N-group. Either

1. $N \cong \langle F^{n \times n}, +, \cdot \rangle$, $A \cong F^n$ for some $n \in \mathbb{N}$, some field F, or
2. $N \cong \langle \{f \in A^A \mid f(0) = 0\}, +, \circ \rangle$.

N-groups of height 2

Question (E. Aichinger)

Assume A is a compatible, faithful N-group with exactly 1 non-trivial, proper N-subgroup B.
Determine N and A.

Example

$\langle \mathbb{Z}_{p^2}, +, \cdot \rangle$ is compatible, faithful on $\mathbb{Z}_{p^2}$ with N-subgroup $p\mathbb{Z}_{p^2}$.

TFAE for a faithful N-group A:

1. N is compatible on A.
2. N is the 0-symmetric part of $\mathrm{Pol}_1(\langle A, +, \{A \to A,\ x \mapsto n * x \mid n \in N\}\rangle)$.

$\mathbb{Z}_{p^2}$

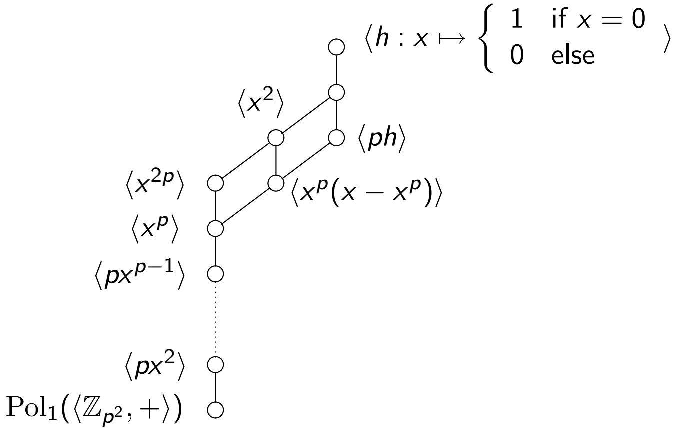
Theorem (cf. Bulatov, '02)

For p an odd prime there are exactly $p+6$ subnear-rings of $\langle \mathbb{Z}_{p^2}^{\mathbb{Z}_{p^2}}, +, \circ \rangle$ that contain $\mathrm{Pol}_1(\langle \mathbb{Z}_{p^2}, + \rangle) = \{x \mapsto ax + b \mid a, b \in \mathbb{Z}_{p^2}\}$.

Corollary

For p an odd prime there are exactly $p+5$ non-isomorphic, 0-symmetric near-rings N with 1 that have $\mathbb{Z}_{p^2}$ as faithful, compatible N-group with N-subgroup $p\mathbb{Z}_{p^2}$.

$p+6$ near-rings containing $\mathrm{Pol}_1(\langle \mathbb{Z}_{p^2}, +\rangle)$



Structure of expanded groups

$\langle A, +, -, 0\rangle \dots$ group
$F \dots$ finitary operations on A
$\mathbf{A} := \langle A, +, -, 0, F\rangle \dots$ expanded group

Definition

$I \lhd \langle A, +\rangle$ is an *ideal* of $\mathbf{A}$ if
$\forall f \in F_n \ \forall a, b \in A^n : a \equiv_I b \ \Rightarrow f(a) \equiv_I f(b)$.

Definition (Scott, cf. Freese, McKenzie, 1987)

The *commutator* $[I, J]$ of ideals I, J of $\mathbf{A}$ is the ideal generated by
$$\{p(i,j) \mid i \in I, j \in J, p \in \mathrm{Pol}_2(\mathbf{A}), \quad p(x,0) = p(0,x) = 0 \ \forall x \in A\}.$$

Reduction using commutators

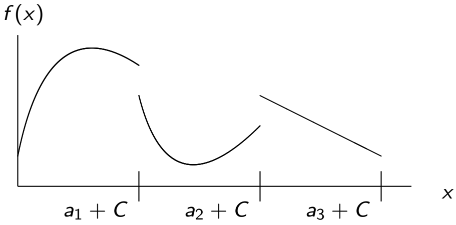
Lemma

Let $\mathbf{A}$ be a finite expanded group with ideals I, J such that $[I, J] = 0$, let $f \in \mathrm{Pol}_1(\mathbf{A})$. Then

$$f(a - b + c) = f(a) - f(b) + f(c) \quad \forall a \equiv_I b, b \equiv_J c.$$

Lemma (PM, unpublished '08)

Let $\mathbf{A}$ be a finite expanded group, let I be a minimal ideal of $\mathbf{A}$ with centralizer C.
Then $f \in \mathrm{Pol}_1(\mathbf{A})$ iff $f_I \in \mathrm{Pol}_1(\mathbf{A}/I)$ and $\forall a \in A \; \exists p \in \mathrm{Pol}_1(\mathbf{A})$: $f|_{a+C} = p|_{a+C}$.



Expansions of $\mathbb{Z}_{p^2}$

Let **A** be an expansion of $\langle \mathbb{Z}_{p^2}, + \rangle$ with ideal $B := p\mathbb{Z}_{p^2}$.
Commutators may take the following values:

$[B, B]$	$[A, B]$	$[A, A]$	
B	B	A	non-abelian–by–non-abelian
B	B	B	non-abelian–by–abelian
0	B	A	abelian-by-non-abelian, trivial center
0	B	B	abelian-by-abelian, trivial center
0	0	A	central-by-non-abelian
0	0	B	**nilpotent, non-abelian**
0	0	0	abelian

Nilpotent case

Lemma

Let p be an odd prime, $B := p\mathbb{Z}_{p^2}$, let $N \leq \mathbb{Z}_{p^2}^{\mathbb{Z}_{p^2}}$. TFAE:

1. $\langle \mathbb{Z}_{p^2}, +, N \rangle$ is nilpotent with center B.
2. $N \subseteq \mathrm{Pol}_1(\langle \mathbb{Z}_{p^2}, +, \cdot \rangle)$, every $f \in N$ is affine modulo B and satisfies

$$f(x + pz) = f(x) - f(0) + f(pz) \ \forall x, z \in \mathbb{Z}_{p^2}.$$

3. Every $f \in N$ is of the form

$$f(x) = ax^p + bx + c + pg(x)$$

for some $a, b, c \in \mathbb{Z}_{p^2}$ and $g \in \mathrm{Pol}_1(\langle \mathbb{Z}_{p^2}, +, \cdot \rangle)$ with $\deg g < p$.

$\mathbb{Z}_{pq}$

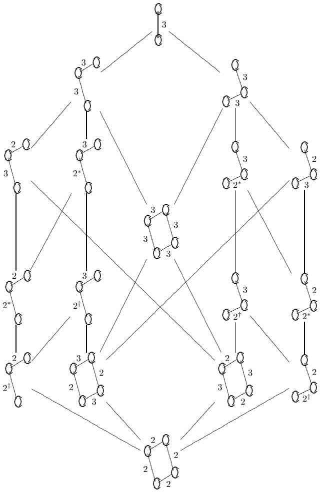
Theorem (Aichinger, PM, '07)

For distinct, odd primes p, q there are exactly 17 subnear-rings of $\langle \mathbb{Z}_{pq}^{\mathbb{Z}_{pq}}, +, \circ \rangle$ that contain $\mathrm{Pol}_1(\langle \mathbb{Z}_{pq}, + \rangle)$.

Corollary

For distinct, odd primes p, q there are exactly 6 non-isomorphic, 0-symmetric near-rings N with 1 that have $\mathbb{Z}_{pq}$ as faithful, compatible N-group with $p\mathbb{Z}_{pq}$ as unique non-trivial, proper N-subgroup.

17 near-rings containing $\mathrm{Pol}_1(\langle \mathbb{Z}_{pq}, +\rangle)$



Bibliography

- E. Aichinger and PM. Polynomial clones on groups of order *pq*. *Acta Math. Hungar.*, 114(3):267–285, 2007.
- A. A. Bulatov. Polynomial clones containing the Mal'tsev operation of the groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$. *Mult.-Valued Log.*, 8(2):193–221, 2002.
- R. Freese and R. N. McKenzie. *Commutator Theory for Congruence Modular Varieties*, volume 125 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, 1987.