1 PRIMENESS IN THE GENERALISED GROUP NEAR-RING

Let R be a right near-ring with identity 1, G be a (multiplicatively written) group with identity e, and let R^G denote the cartesian direct sum of $\mid G \mid$ copies of $(R,+)$ indexed by the elements of G. Then $M(R^G)$ is the right near-ring of all mappings of the group R^G into itself. Now, for $r \in R$ and $g \in G$, let $[r,g]$ denote the function in $M(R^G)$ defined by:

$$([r,g](\mu))(h) = r\mu(hg) \text{ for all } \mu \in R^G, h \in G.$$

Definition 1 *(Group Near-ring)*
The subnear-ring of $M(R^G)$, generated by the set $\{[r,g] : r \in R, g \in G\}$, will be called the group near-ring constructed from R and G and will be denoted by $R[G]$.

Now let M be a left R-module. Then it is also well known that the set of all mappings from the group M^G into itself is a right near-ring with addition defined pointwise and multiplication defined by composition of functions. Now, for $r \in R$, $g \in G$ and $\mu \in M^G$, let the function $[r,g] : M^G \to M^G$ be defined by $([r,g]\mu)(h) = r\mu(hg)$ where $h \in G$. (Note that, since M is an R-module, $r\mu(hg) \in M$).

Definition 2 *The generalised group near-ring, denoted by $R[G : M]$, is defined to be the subnear-ring of all mappings from M^G to M^G generated by the set $\{[r,g] : r \in R, g \in G\}$.*

Remark 3 *We note that:*

(a) *If we let M be the special case, $M = {}_RR$, then $R[G : M]$ is simply the group near-ring, $R[G]$, as defined by Le Riche, Meldrum and Van der Walt.*

(b) *If $A : M^G \longrightarrow M^G$, $B : M^G \longrightarrow M^G$ are elements of $R[G : M]$ and $\mu \in M^G$, then under the natural operations:*

$$(A+B)\mu = A\mu + B\mu$$
$$(AB)\mu = A(B\mu)$$

M^G is an $R[G : M]$-module

Definition 4 *Let P be a subset of the R-module M. Then we define the subset, P^* of $R[G:M]$ by:*

$$P^* = \{A \in R[G:M] : A\mu \in P^G \text{ for all } \mu \in M^G\}.$$

Proposition 5 *If P is an R-ideal of M, then P^* is an ideal of $R[G:M]$.*

Another way of constructing an ideal in $R[G:M]$ lies in the following definition:

Definition 6 *Let P be an R-ideal of the R-module M. Then we define P^+ to be the ideal in $R[G:M]$ generated by the set: $\{[a,e] : a \in (P:M)_R\}$.*

2 PRIME NEAR-RING MODULES

Definition 7 *Let $P \triangleleft_R M$ such that $RM \not\subseteq P$. Then P is called:*

(a) *0-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all ideals, A of R, and all R-ideals, B of M.*

(b) *1-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all left ideals, A of R, and all R-ideals, B of M.*

(c) *2-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all left R-subgroups, A of R, and all R-submodules, B of M.*

(d) *3-prime if $rRm \subseteq P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.*

(e) *completely prime (c-prime) if $rm \in P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.*

Definition 8 *M is said to be a ν-prime ($\nu = 0,1,2,3,c$) R-module if $RM \neq 0$ and 0 is a ν-prime R-ideal of M.*

If $P \triangleleft_R M$, then we recall that $\widetilde{P} = (P:M)$ is an ideal of R. Now, if P is a ν-prime ($\nu = 2,3,c$) R-ideal then does this imply that $\widetilde{P}$ is also ν-prime? We investigate this in the propositions that follow. The case $\nu = 0$ is treated separately.

Proposition 9 *Let P be an R-ideal of M. Then:*

(a) P is a 2-prime R-ideal of M implies that $\widetilde{P}$ is a 2-prime ideal of R.

(b) P is a 3-prime R-ideal of M implies that $\widetilde{P}$ is a 3-prime ideal of R.

(c) P is a completely prime R-ideal of M implies that $\widetilde{P}$ is a completely prime ideal of R.

Proposition 10 *Let P be a 0-prime R-ideal of a monogenic (tame) R-module M. Then $\widetilde{P}$ is a 0-prime ideal of R.*

3 PRIME RELATIONS BETWEEN R, M AND $R(G:M)$

Proposition 11 *If P is an R-ideal of a 2-multiplication module M such that P^* is a 3-prime ideal of $R[G:M]$, then P is a 3-prime R-ideal of M.*

Corollary 12 *If P is an R-ideal of a 2-multiplication module M such that P^* is a 3-prime ideal of $R[G:M]$, then $\widetilde{P}$ is a 3-prime ideal of the near-ring R.*

Proposition 13 *Let $\mathbb{P}$ be an ideal of $R[G:M]$. Then $\mathbb{P}$ is 2-prime if and only if $\mathbb{P}$ is 3-prime.*

Corollary 14 *If P is an R-ideal of a 2-multiplication module M such that P^* is a 2-prime ideal of $R[G:M]$, then P is a 2-prime R-ideal of M.*

Corollary 15 *If P is an R-ideal of a 2-multiplication module M such that P^* is a 2-prime ideal of $R[G:M]$, then $\widetilde{P}$ is a 2-prime ideal of the near-ring R.*

Proposition 16 *Let P be an R-ideal of M and assume that P^* is a 0-prime ideal of $R[G;M]$. Then P is a 0-prime R-ideal of M.*

Corollary 17 *Let P be an R-ideal of a monogenic (or tame) R-module M and assume that P^* is a 0-prime ideal of $R[G;M]$. Then $\widetilde{P}$ is a 0-prime ideal of R.*

Thus far we have shown that if P is an R-ideal of M (a 2-multiplication module in some cases) such that P^* is a ν-prime ideal in $R[G:M]$, then P is a ν-prime R-ideal of M, where $\nu = 0, 2, 3$. In the next part of this section, we investigate the reverse situation of some of these implications by considering the primeness of $R[G:M]$ under certain conditions imposed on R and/ or M.

Proposition 18 *Let R be any near-ring, M a nonzero R-module and G be a cyclic group of order n. Then $R[G:M]$ is not completely prime.*

Proposition 19 *Let M be an R-module with $\mid M \mid \geqslant 2$. Suppose that R contains a nonzero element c with the property that $cm_1 + cm_2 = cm_2 + cm_1$ for all $m_1, m_2 \in M$. If G is finite with $\mid G \mid \geq 2$, then $R[G:M]$ is not 3-prime.*

Corollary 20 *If M is an abelian R-module and G is a finite group, then $R[G:M]$ is not 3-prime.*

Corollary 21 *If M is an abelian R-module and G is a finite group, then $R[G:M]$ is not completely prime*

Corollary 22 *If M is an abelian R-module and G is a finite group, then $R[G:M]$ is not 2-prime.*

In the preceding theorems, we demonstrated many negative results. However, we now present a positive result which could have a great impact on future research on group near-rings.

Theorem 23 *Let R be a near-field, $M = {}_RR$ and G be an ordered group. Then $R[G:M]$ is 2-prime.*

Corollary 24 *If R is a near-field, $M = {}_RR$ and G is an ordered group, then $R[G:M]$ is ν-prime for $\nu = 0, 2, 3$.*