Maximal Endomorphism Semirings

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Nearring Conference, 2009

Introduction I

- Suppose first that $(A, +)$ is an abelian group
- Then $Map(A) := \{f : A \to A\}$ is a commutative nearring under pointwise addition and composition of functions
- Let $M_0(A) := \{f : A \to A | f(0) = 0\}$, the nearring of zero preserving functions on A,
- and $End(A) := \{f : A \to A | f(x + y) = f(x) + f(y)\}$

Introduction II

Question ??

When will $End(A)$ be maximal as a ring in $M_0(A)$?

"E-locally cyclic abelian groups and maximal nearrings of mappings"- Kreuzer and Maxson, Forum Mathematica (2006)

Definition

We say the abelian group $A = (A, +)$ is <u>E-locally cyclic</u> if for each $a, b \in A$ there exists a $c \in A$ and $\alpha, \beta \in End(A)$ such that $\alpha(c) = a$ and $\beta(c) = b$.

Introduction III

Theorem

If A is an E-locally cyclic abelian group, then $End(A)$ is a maximal subring of $M_0(A)$.

Proof.

- Let $End(A) \subseteq R$, where R is a subring in $M_0(A)$
- Now let $\sigma \in R$ and $a, b \in A$
- Since A is E-locally cyclic, there exists some $c \in A$ and $\alpha, \beta \in End(A)$ such that $\alpha(c) = a$ and $\beta(c) = b$
- Since α, β are also in R and since R is a ring, we have $\sigma(\alpha + \beta) = \sigma\alpha + \sigma\beta$
- Thus $\sigma(a + b) = \sigma(\alpha(c) + \beta(c)) = \sigma((\alpha + \beta)(c)) = (\sigma\alpha + \sigma\beta)(c) = \sigma\alpha(c) + \sigma\beta(c)$, so $\sigma \in End(A)$ giving $R = End(A)$.

Introduction IV

Definition

We say the abelian group $A = (A, +)$ is <u>torsion (periodic)</u> if for each $a \in A$ there exists a positive integer n such that $na = 0$.

- Kreuzer and Maxson (2006) showed that every torsion group is E-locally cyclic
- Every finite group is torsion
- E-locally cyclic implies that $End(A)$ is a maximal subring of $M_0(A)$
- Thus torsion $\Rightarrow$ E-locally cyclic $\Rightarrow$ maximality.

Monoids I

- Let $(M, +, 0)$ be an abelian monoid
- We call $Map(M)$ a near-semiring since, in general, composition is one-sided distributive over $+$
- Since M is abelian, the sum of two endomorphisms is an endomorphism, so $End(M)$, under pointwise addition and function composition, is a semiring contained in $Map(M)$
- Here we can have that $End_0(M) \subset End(M)$.

Monoids II

Example

- Let $\mathbb{N}_n := \{0, 1, 2, \ldots, n\}$, n a positive integer, and let $+$ be defined by $x + y = max\{x, y\}$
- Then $(\mathbb{N}_n, +, 0)$ is a commutative monoid
- And for any $m \in \mathbb{N}_n \backslash \{0\}$, the function $k_m : \mathbb{N}_n \to \mathbb{N}_n$ given by $k_m(x) = m$, $x \in \mathbb{N}_n$ is in $End(\mathbb{N}_n) \backslash End_0(\mathbb{N}_n)$.

Question ??

When $End_0(M)$ is maximal as a semiring in $Map(M)$?

Monoids III

Proposition

For an abelian monoid $M = (M, +, 0)$, $End_0(M) = End(M)$ if and only if 0 is the only idempotent in M.

Proof.

- We always have that $End_0(M) \subseteq End(M)$
- Suppose there is another idempotent in M, then as in the previous example we can construct a map in $End(M)$, but not in $End_0(M)$
- This contradicts the assumption that $End_0(M) = End(M)$
- On the other hand, if 0 is the only idempotent and $f \in End(M)$, then $f(0) = f(0 + 0) = f(0) + f(0)$ so $f(0) = 0$.

Monoids IV

Definition

If a and b are elements of a semigroup S, we say that a divides b, if there exists an $x \in S$ such that $ax = b$.

Definition

A commutative semigroup S is said to be archimedean if, for any two elements of S, each divides a power of the other.

Definition

A commutative idempotent semigroup is called a semilattice.

Monoids V

Theorem

Every commutative semigroup S is uniquely expressible as a semilattice Y of archimedian semigroups S_α $(\alpha \in Y)$.

- Let $M = \cup A_\alpha$, $\alpha \in Y$ be the decomposition of M into its Archimedian components
- Suppose M is periodic
- Then for each $a_\alpha \in A_\alpha$, the semigroup generated by a_α contains an idempotent
- Hence if $End_0(M) = End(M)$ then M is an Archimedean semigroup with an idempotent
- Then we have that each element of M has an additive inverse (Theorem),
- so we find that M is an abelian group.

Monoids VI

Theorem

Let M be a periodic commutative monoid. Then $End_0(M)$ is a maximal semiring in $Map(M)$ if and only if M is an abelian group.

Semigroups I

We will now look at situations where $End(S)$ is a maximal semiring in $Map(S)$, where S is an abelian semigroup.

Definition

We say the semigroup $S = (S, +)$ is E-locally cyclic if for each $a, b \in S$ there exists a $c \in S$ and $\alpha, \beta \in End(S)$ such that $\alpha(c) = a$ and $\beta(c) = b$.

As with abelian groups, we have that

Proposition

If S is an E-locally cyclic commutative semigroup, then $End(S)$ is a maximal semiring in $Map(S)$.

Semigroups II

Corollary

If S is a semilattice, that is, a commutative idempotent semigroup, then S is E-locally cyclic.

Proof.

- Consider the mapping $k_s(x) : S \to S$ defined by $k_s(x) = s$ for all $x \in S$
- It is easy to verify that $k_s \in End(S)$
- Now let $a, b \in S$
- We have that $k_a(c) = a$ and $k_b(c) = b$, hence S is E-locally cyclic.

□

Semigroups III

Example

- Let $R := (R, +, \cdot)$ be a semiring
- For each $n \in R$, the map $\lambda_n : R \to R$, $\lambda_n(x) = nx$ is an endomorphism of the commutative semigroup $(R, +)$
- If, in addition, R has a multiplicative identity, 1, then $\lambda_n(1) = n \cdot 1 = n$, so $(R, +)$ is E-locally cyclic

Generalising, we have

Proposition

Let $S = (S, +)$ be a commutative semigroup. If S has a left distributive multiplication, $\cdot$, and a right distributive identity, 1, then S is E-locally cyclic.

In closing

- More can be done for certain specialised semigroups
- In our paper we look specifically at commutative Clifford Semigroups

Definition

A Clifford semigroup is a strong semilattice of groups. Thus

- $S = \cup_{\alpha \in Y} G_\alpha$, where the G_α are disjoint abelian groups
- And Y is a semilattice
- Then for each pair $\{\alpha, \beta\} \in Y$, with $\alpha \geq \beta$ there exists a group homomorphism $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ such that
  - $\phi_{\alpha,\alpha} = id$ on G_α;
  - $\phi_{\beta,\psi}\phi_{\alpha,\beta} = \phi_{\alpha,\psi}$ for $\alpha, \beta, \psi \in Y$, $\alpha \geq \beta \geq \psi$
- Then the operation $+$ on S is given by
- $a_\alpha + b_\beta = \phi_{\alpha,\alpha\beta}(a_\alpha) + \phi_{\beta,\alpha\beta}(b_\beta)$, $a_\alpha \in G_\alpha$, $b_\beta \in G_\beta$.

Some Reading

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- J.S. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science,* Longman Scientific & Technical, First Edition, 1992.
- J.M. Howie, *An Introduction to Semigroup Theory,* Academic Press Inc. (London), 1976.
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