Primeness in near-rings

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All near-rings are **right near-rings**. We will use $\mathcal{R}, \mathcal{N}$ and $\mathcal{N}_0$ denote the variety of all **rings**, **near-rings** and **zero-symmetric near-rings** respectively.

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Now, since R is a zero symmetric near-ring we have $rRr = (0)$ with $0 \neq r$ and R does not satisfy condition (2)

NOTATION

For $K \subseteq R$, $< K|_R$, $| K >_R$, $< K >_R$, $< K]_R$ and $[K >_R$ denote the **left ideal**, **right ideal**, **two-sided ideal**, **left** R-**subgroup** and **right** R-**subgroup** generated by K in R respectively. If it is clear in which near-ring we are working, the subscript R will be omitted.

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Also $K \lhd_l R$, $K \lhd_r R$, $K \lhd R$ and $K \lhd_R R$ symbolize that K is a **left ideal, right ideal**, **two-sided ideal** or a **left** R**-subgroup** of R.

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types $r1$ and $r2$ are due to Birkenmeier [1].

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This is in sharp contrast to the ring case where $0-$prime and $3-$prime are equivalent.

Different prime radicals

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Definition

Let $R \in \mathcal{N}$. Then $\mathfrak{P}_v(R) = \cap\{P \lhd R : P \text{ is } v\text{-prime}\}$is the v-**prime radical** of R for $v \in \{0, 1, r1, 2, r2, 3\}$.

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Clearly all the v-prime radical maps $\mathfrak{P}_v$ are Hoehnke radicals.

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Kurosh Amitsur Radical

If $I \triangleleft R$ then a Hoehnke radical gives no information on the relationships between $\rho(R)$ and $\rho(I)$.
But this is, amongst others, what the general theory of radicals is all about: Given a near-ring R, then it should provide some information on the relationship between $\rho(R)$ and the radicals of near-rings related to R e.g., homomorphic images, ideals, extensions, etc.
The following relationships between the radicals of a near-ring and its ideals play an important role in the general theory of radicals:

Definition

An H-radical ρ is:

(H3) **complete** if $\rho(I) = I \triangleleft R$ implies $I \subseteq \rho(R)$;

(H4) **idempotent** if $\rho(\rho(R)) = \rho(R)$;

(H5) **ideal-hereditary** if $\rho(I) = I \cap \rho(R)$.

If ρ is an H-radical which is **idempotent and complete**, then it is called a **Kurosh-Amitsur** *(KA) radical* map.

Which of the prime radicals are KA-radicals

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There are examples to show that $\mathfrak{P}_1(\mathfrak{P}_1(R)) \neq \mathfrak{P}_1(R)$ and therefore $\mathfrak{P}_1$ is not idempotent. Thus $\mathfrak{P}_1$ is not a KA-radical.

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What the situation is for $\mathfrak{P}_2$ and $\mathfrak{P}_3$ in $\mathcal{N}_0$ is not known.

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It is easy to check that an equiprime near-ring is zero-symmetric and 3-prime.

Ideal Hereditary KA prime radical

Let $\mathfrak{P}_e$ denote the **equiprime radical map**:

$$\mathfrak{P}_e(R) = \cap\{I \triangleleft R : R/I \text{ equiprime}\}$$

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then $\mathfrak{P}_e$ is an **ideal-hereditary** *KA*-**radical** in the variety of all near-rings i.e., $\mathfrak{P}_e(R) \cap I = \mathfrak{P}_e(I)$ for every $I \triangleleft R \in \mathcal{N}$.

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- For any group G, the simple near-ring with identity $\mathcal{M}_0(G)$ is equiprime.
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If we could replace invariant subgroup by two-sided ideal in the theorem above we would have a positive answer to the above question

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Hence $1 \in R = A$ from which $x = y$ follows and R is equiprime

Holcombe [14] defined an $R-$ group G to be of type 3 if $RG \neq 0$, G has no nontrivial $R-$subgroups, and for $g_1, g_2 \in G$, $rg_1 = rg_2$ for all $r \in R$, implies $g_1 = g_2$.

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Example

Let R be the near-ring built on any cyclic group of uneven prime order with multiplication given by $ab = \begin{cases} a & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$

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This near-ring is 3-prime, 2-primitive, satisfy the descending chain condition on $R-$subgroups and zero-symmetric but it is not i-equiprime.

2-Primal Near-rings

An ideal I of a near-ring R is called a *completely prime deal* of R if whenever $ab \in I$, then $a \in I$ or $b \in I$.

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The intersection of all of the completely prime ideals of R, denoted herein by $\mathfrak{P}_c(R)$, is the *completely prime radical* of R.
Birkenmeier et al investigated conditions under which a 0-prime ideal is completely prime and conditions for which **every** 0-prime ideal in a near-ring is completely prime. They introduced the concepts of 2- **primal near-rings** and 2-**primal ideals**.

2-Primal Near-rings

Definition

An ideal I of R is a **2-primal ideal** of R if $\mathfrak{P}_0(R/I) = N(R/I)$. ($N(R)$ denotes the set of **nilpotent elements** of the near-ring R). If the **zero ideal** of R is a 2-primal ideal, then R is a **2-primal near-ring**. (This is equivalent to $\mathfrak{P}_0(R) = N(R)$).

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Some **examples** of 2-primal near-rings which immediately come to mind are those which are **commutative, anti-commutative** ($ab = -ba$ for all $a, b \in R$), **nilpotent**, or **reduced**. (We say a subset of a near-ring is *reduced* if it contains no nonzero nilpotent elements).

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These are near-rings R such that every $0-$prime ideal is a completely prime ideal.
The class of all these near-rings will be denoted by $\mathfrak{R}^2$ and $\mathfrak{R}_0^2 = \mathfrak{R}^2 \cap \mathcal{N}_0$.From this it is clear that if $R \in \mathcal{N}_0$, Then $R \in \mathfrak{R}_0^2$ if and only for every ideal I of R, R/I is 2-primal.

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- **QUESTION : Can we a define a notion of 2-primal for near-rings for which the corresponding class $\mathfrak{R}_0^2$ will be a KA radical class .**

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As in [6] we define class pairs and radical pairs as follows.

Definition

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are classes of near-rings and ρ_1 and ρ_2 ideal maps. Then

1. $(\mathcal{M}_1 : \mathcal{M}_2) = \{R : \text{for } I \lhd R,\ R/I \in \mathcal{M}_2 \Rightarrow R/I \in \mathcal{M}_1\}$,
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Theorem

Let ρ_1 and ρ_2 be H-radical maps associated with $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Then $\mathcal{M}_1 \cap \mathcal{S}_{\rho_2} \subseteq \mathcal{M}_2$ implies $(\mathcal{M}_2 : \mathcal{M}_1) = (\rho_1 : \rho_2)$.

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Let $\mathcal{P}_i$ *denote the class of* $i-$*prime near-rings and* $\mathfrak{P}_i$ *the associated H-radical,* $i = 0, 2, 3$ *and* c. *Then* $(\mathcal{P}_i : \mathcal{P}_0) = (\mathfrak{P}_0 : \mathfrak{P}_i)$, $i = 2, 3$ *and* c.

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Remark: From this we have $\mathfrak{R}_0^2 = (\mathcal{P}_c : \mathcal{P}_0) = (\mathfrak{P}_0 : \mathfrak{P}_c)$.

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This shows $(\mathfrak{P}_3 : \mathfrak{P}_c) = (\mathcal{P}_c : \mathcal{P}_3)$ forms a KA-radical class since $(\mathcal{P}_c : \mathcal{P}_3)$ is clearly also homomorphically closed.

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Then $(\mathfrak{P}_3 : \mathfrak{P}_c) = (\mathcal{P}_c : \mathcal{P}_3) = (\mathcal{P}_c^s : \mathcal{P}_3)$ where $\mathcal{P}_c^s$ is the class of all completely semiprime near-rings and $\mathcal{P}_c^s \cap \mathcal{P}_3 \subseteq \mathcal{P}_c$.
By [12, Lemma 4.3], $\mathcal{P}_3$ is essentially closed and it follows from [5] that $\mathcal{P}_3$ is an hereditary class.
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- In [4] it was shown that the class $\mathfrak{R}_0^2$, which is equal to $(\mathfrak{P}_0 : \mathfrak{P}_c)$, is not a KA-radical in general.
- From this, it follows that the near-rings R for which $\mathfrak{P}_3(R) = \mathfrak{P}_c(R)$ is a more appropriate generalization of the notion of 2-primal from rings to near rings.

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This contradicts the fact that $(\mathfrak{P}_3 : \mathfrak{P}_c) = (\mathcal{P}_c : \mathcal{P}_3)$ forms a KA-radical class and we conclude $\mathcal{P}_3^s \neq \mathcal{S}_{\mathfrak{P}_3}$.

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Hence there are near-rings with 3-semi prime ideals which can not be written as the intersection of 3-prime ideals.
This gives an answer in the negative to a long standing open question posed by Gordon Mason.

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Hence if R is a near-ring then $\mathbb{N}(R)$ i.e., the sum of all nil ideals of R is equal to $s(R)$ the intersection of all the s-prime ideals of R (all ideals I such that R/I is an s-prime near-ring).

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He mentioned that the proof of this result is essentially that given for rings by Divinsky, see [9, page 147].

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In [5] Birkenmeier et al called an ideal I of the near-ring R **nilprime** if I is a 0-prime ideal and $\mathbb{N}(R/I) = 0$ i.e., R/I has no nonzero nil ideals. They then gave a self-contained proof within near-ring theory of the result "that the nil radical $\mathbb{N}(R)$ of the near-ring R is equal to the intersection of all the 0-prime ideals P of R such that R/P has no nonzero nil ideals".mentioned by Kaarli.

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In this talk we introduce another notion of an s-prime near-ring which coincides with the notion of nilprime.

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The subset M of the near-ring R is called an $m-$**system** if for every $a, b \in M$ there exists $c \in< a >< b >$ such that $c \in M$.

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Clearly an $s-$system is an $m-$system and also an $ss-$system.

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An ideal Q of the near-ring R is an $s-$**prime**, **prime (0-prime)**, $s-$**semiprime** or **semiprime (0-semiprime)** ideal if $\mathcal{C}_R(Q)$ is an $s-$system, $m-$system, $ss-$system or a $sp-$system respectively.

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Theorem

The $s-$radical $s(R)$ of the near ring R is equal to the intersection of all the $s-$prime ideals of R and coincides with the upper nil radical $\mathbb{N}(R)$ of R

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We now show that in the case of near-rings this give rise to a number of nonequivalent nilradicals.

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This near-ring is not $1-$nilprime since R is not $1-$prime because if $I = (0 : G \backslash H)$ then I is a left ideal of R and $AI = (0 : H)(0 : G/H) = 0$.

1-nilprime but not 2-nilprime

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Furthermore, for every $0 \neq x \in G \backslash H$ we have

$x^n = x^{n-2} \cdot (x \cdot x) = x^{n-1} = \cdots = x \neq 0.$

Hence $\mathbb{N}((G, +, \cdot)) = 0$

Thus $(G, +, \cdot)$ is a $1-$nilprime near-ring but not a $2-$nilprime near-ring.

2-nilprime but not 3-nilprime

Example

Let R be the near-ring on $\mathbb{Z}_3 = \{0, 1, 2\}$ multiplication defined by:

$$a \cdot b = \begin{cases} a & \text{if} \quad b = 2 \\ 0 & \text{if} \quad b \neq 2 \end{cases} .$$

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Furthermore we have $2^n = 2$ for every $n \in \mathbb{N}$.

Thus R is a $2-$nilprime near-ring but not a $3-$nilprime near-ring.

3-nilprime but not equi-nilprime

Example

If $(R, +)$ is any cyclic group of prime order p $(p > 2)$, define a near-ring multiplication on R by:

$$ab = \begin{cases} a & \text{if} \quad b \neq 0 \\ 0 & \text{if} \quad b = 0 \end{cases}$$

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Then R is a near-ring which is $3-$nilprime but not equi-nilprime.

Relation between radicals

Example

Near-ring number 17 defined on S_3 [18] is an example of an $r1-$nilprime near-ring which is not $r2-$nilprime and near-ring number 20 on S_3 [18] is an example of an $r2-$nilprime near-ring which is not $r3-$nilprime.

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If R is any near-ring and $\rho_{n_i}(R)$ denotes the $H-$radical determined by the class of $i-$nilprime near-rings, then

- $\mathbb{N}(R) = \rho_{n_0}(R) \subsetneq \rho_{n_1}(R) \subsetneq \rho_{n_2}(R) \subsetneq \rho_{n_3}(R) \subsetneq \rho_{n_e}(R)$ and

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$$[a]^i[b]^i = \begin{cases} <a><b> & \text{for } i = 0 \\ <a|<b| & \text{for } i = 1 \\ |a>|b> & \text{for } i = r1 \\ [a>_R [b>_R & \text{for } i = r2 \\ <a]_R <b]_R & \text{for } i = 2 \\ aRb & \text{for } i = 3 \end{cases}$$

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NOTE:

An ideal Q of R is i-prime, $i \in \{0, 1, r1, 2, r2, 3\}$, if for $a, b \in R$, $[a]^i[b]^i \subseteq Q$ implies $a \in Q$ or $b \in Q$.

i-nilprime ideals and systems

Definition

A subset T of the near-ring R is called a **complete system** if $a^n \in T$ for every $a \in T$ and every $n \in \mathbb{N}$.

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Theorem

An ideal Q of R is $i - s-$prime, $i \in \{0, 1, r1, 2, r2, 3\}$, if and only if Q is i-nilprime if and only if $\mathcal{C}_R(Q)$ is an n_i-system.

s-equiprime near-rings

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[21, page 258] A near-ring R is s-equiprime if it contains a nonempty multiplicative closed set S with $0 \notin S$ such that $0 \neq a \in R$ and $T_R(a,x,y) \cap S = \varnothing$ implies $x = y$ $(x, y \in R)$ where $T_R(a,x,y) = \{$all finite sums $\sum_i r_i(as_ix - as_iy)k_i$ with $r_i, s_i, k_i \in R\}$. In such a case S is called the kernel of R.

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Theorem

Every $s-$equiprime near-ring is equi-nilprime.

Question

We know that **equiprime radical map**:
$\mathfrak{P}_e(R) = \cap\{I \triangleleft R : R/I \text{ equiprime}\}$ is an **ideal-hereditary** KA**-radical map** in the variety of all near-rings i.e., $\mathfrak{P}_e(N) \cap I = \mathfrak{P}_e(I)$ for every $I \triangleleft N \in \mathcal{N}$

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- If R is a near-ring we know that $\rho_{n_e}(R) \subseteq s_e(R) = \cap\{I \triangleleft R : R/I \text{ } s\text{-equiprime}\}$. When will $\rho_{n_e}(R) = s_e(R)$?

Near-ring modules

Let R be a near-ring and let M, be any left R-module and P a subset of R.If P is an $R-$ideal ($R-$submodule) of M we denote it by $P \lhd_R M$ ($P \leq_R M$).

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We attempt to generalize the various notions of primeness that were defined in R to the module M.

Prime near-ring modules

Definition

Let $P \triangleleft_R M$ such that $RM \not\subseteq P$. Then P is called:

- **0-prime** if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all ideals, A of R, and all R-ideals, B of M.

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- **3-prime** if $rRm \subseteq P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.
- **completely prime** (c-prime) if $rm \in P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.

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M is said to be a ν-prime $(\nu = 0, 1, 2, 3, c)$ R-module if $RM \neq 0$ and 0 is a ν-prime R-ideal of M.

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In general, we cannot distinguish between 0-prime and 1-prime near-ring modules. Thus 1-prime modules were omitted from further investigations.

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Let $P \lhd_R M$. Then the following are equivalent:

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3. *For all $a \in R$ and $b \in M$ such that $a[b]_R \subseteq P$, it follows that $aM \subseteq P$ or $b \in P$. (Here $[b]_R$ is the submodule of M generated by b).*

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4. *For all R-submodules N of M such that $P \subset N$, we have that $(P : M) = (P : N)$.*

In a similar way we can construct and prove equivalent definitions for 0-prime and 1-prime R-ideals.

Theorem

Let P be an R-ideal of M. Then the following are equivalent:

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3. *For all $a \in R$ and $b \in M$ such that $a\langle b\rangle_R \subseteq P$, we have that $aM \subseteq P$ or $b \in P$. (Here $\langle b\rangle_R$ is the R-ideal of M generated by b).*

In a similar way we can construct and prove equivalent definitions for 0-prime and 1-prime R-ideals.

Theorem

Let P be an R-ideal of M. Then the following are equivalent:

1. *P is a 0-prime (or 1-prime) R-ideal.*
2. *For all $a \in R$ and for all R-ideals B of M such that $aB \subseteq P$, we have that $aM \subseteq P$ or $B \subseteq P$.*
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4. *For all R-ideals N of M such that $P \subset N$, we have that $(P : M) = (P : N)$.*

Corollary

An R-module M is:

1. *0-prime if and only if for all non-zero R-ideals N of M, it follows that* $(0 : M) = (0 : N)$.

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1. *0-prime if and only if for all non-zero R-ideals N of M, it follows that $(0:M)=(0:N)$.*
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Theorem

Let M be an R- module and $P \lhd_R M$. *Then the following are equivalent:*

1. P *is* 3- *prime and* $(P : m) \lhd R$ *for every* $m \in M \setminus P$.

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Let $P \lhd_R M$. *Then the following are equivalent:*

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Theorem

Let $P \lhd_R M$. Then the following are equivalent:

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Theorem

Let $P \lhd_R M$. *Then the following are equivalent:*

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Let $P \lhd_R M$. *Then the following are equivalent:*

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Let $P \lhd_R M$. Then the following are equivalent:

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Theorem

Let $P \lhd_R M$. Then P is completely prime $\Rightarrow$ P is 3-prime $\Rightarrow$ P is 2-prime $\Rightarrow$ P is 0-prime.

In general, a 0-prime R-ideal need not be 2-prime and a 2-prime R-ideal need not be 3-prime.

Relation between prime ideals in and prime R-ideals

If $P \lhd_R M$, then we recall that $\widetilde{P} = (P : M)$ is an **ideal** of R.

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- *If M is a tame $R-$module and P be a 0-prime R-ideal of M, then $\widetilde{P}$ is a 0-prime ideal of R.*
- *Let P be a 0-prime R-ideal of a monogenic R-module M. Then $\tilde{P}$ is a 0-prime ideal of R.*

- Now suppose that P is an R-ideal of M such that $(P : M)$ is a ν-prime ideal of R for $\nu = 0, 2, 3$ and c. Does this imply that P is a ν-prime R-ideal of M?

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For the various types of prime R-ideals (modules) we were easily able to prove that if an R-ideal P of M satisfied a certain prime condition, then so did the corresponding ideal $\widetilde{P} = (P : M)$ of R.

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However the converse relation turned out to be problematic in many situations, especially since it is difficult to construct an R-ideal of M by starting with an ideal of R.
To overcome this problem, we now introduce the notion of a multiplication near-ring module.

Multiplication near-ring modules

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Let M be an $R-$module. Then:

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1. *M is called a 0-multiplication module if every R-ideal is multiplication ideal.*
2. *M is called a 2-multiplication module if every R-submodule is multiplication submodule.*
3. *M is called a c-multiplication module if every $m \in M$ is a multiplication element.*

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- *Let P be an R-ideal of a 0-multiplication R-module M such that $\tilde{P}$ is a 0-prime ideal of R. Then P is a 0-prime R-ideal of M.*

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Corollary

Suppose that M is a ν-multiplication R-module $(\nu = 0, 2, c)$. Then M is ν-prime if and only if R is ν-prime. Furthermore, if M is a c-multiplication module, then M is 3-prime if and only if R is 3-prime.

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