Seminearrings of Polynomials over Lattices

July 2009

Introduction

$(S, +, *)$ is a *seminearring* if

- $(S, +)$ is a semigroup
- $(S, *)$ is a semigroup
- $(a + b) * c = a * c + b * c$ for all $a, b, c \in S$

Example: mappings of a semigroup to itself

$(S, +, *)$ is a *seminearfield* if

- $(S, +)$ is a semigroup
- $(S^*, *)$ is a group (S^* is S without its additive zero, if it has one)
- $(a + b) * c = a * c + b * c$ for all $a, b, c \in S$

Example: lattice ordered group

Lattices, functions and polynomials

We investigate lattices $(L, \vee, \wedge)$.
An element $j \in L$ is *join irreducible* if $j = a \vee b \Rightarrow a = j$ or $b = j$. The set of join irreducible elements in a lattice is written $\mathcal{J}(L)$. Similarly we have the set $\mathcal{M}(L)$ of *meet irreducible* elements.
The *dual* L^δ is the lattice with $a \leq b$ in L^δ iff $b \leq a$ in L.

Order preserving functions in one variable over a lattice $(L, \vee, \wedge)$ form two seminearrings, $(O(L), \vee, \circ)$ and $(O(L), \wedge, \circ)$.
The set of *polynomial functions* are the functions generated by the constant maps and the identity map in the set of (order preserving) functions.
Polynomial functions in one variable over a lattice $(L, \vee, \wedge)$ form two seminearrings, $(Pol_1(L), \vee, \circ)$ and $(Pol_1(L), \wedge, \circ)$.

Order polynomial completeness

In general $Pol_1(L) \subseteq O(L)$.
L is called *order polynomial complete* (OPC) if $Pol_1(L) = O(L)$.

Theorem

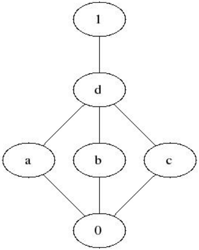
L is OPC iff L is tolerance simple.

A *tolerance* of an algebra (A, F) is a reflexive symmetric binary relation respected by the operations F of an algebra. Equivalently it is a symmetric subalgebra of A^2 including the diagonal $\{(a, a); a \in A\}$.
The collection of tolerances of an algebra form an algebraic lattice. Meet is intersection.
An algebra is *tolerance trivial* if all its tolerances are congruences (e.g. groups) and *tolerance simple* if it has no nontrivial tolerances.
A tolerance of a lattice is a collection of (possibly overlapping) intervals in that lattice.

Examples

Example: M_n, the lantern of width n, is tolerance simple.

Example: The following lattice has the nontrivial tolerance $[0, d] \cup [a, 1]$ which is not a congruence.



Distributive Lattices

Theorem (Birkhoff)

L a finite distributive lattice, then $L \cong \mathcal{O}(\mathcal{J}(L))$

Note also that $\mathcal{J}(L) \cong \mathcal{M}(L)$.

Theorem (Niederle 1982)

L a distributive lattice, then the lattice of meet irreducible tolerances is isomorphic to the lattice of intervals in $\mathcal{J}(L)$

In particular there are always nontrivial tolerances.

Distributive Lattices

Theorem (Mitsch 1970)

The unary polynomials on a finite lattice L are given by $a \vee (b \wedge x)$ for some $a \leq b$.

In particular the polynomial is determined by the images of 0 and 1, i.e. $Pol_1(L) \cong L^{\underline{2}}$. In general we have that $Pol_k(L) \cong L^{\underline{2}^k} = L^{B_k}$.

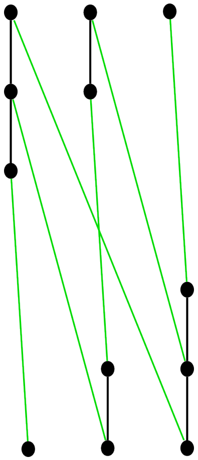
Theorem

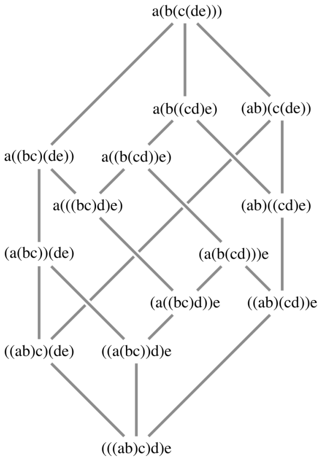
L a finite distributive lattice, then $\mathcal{J}(Pol_k(L)) \cong \mathcal{J}(L) \times B_k$.

Generalizing these results

Given a finite lattice L, define a relational structure $(\mathcal{J}(L), \mathcal{M}(L), E)$ where $E = \not\leq \cap (\mathcal{J}(L) \times \mathcal{M}(L))$, the *M-representation* of L. The transitive closure of $\leq_J \cup \leq_M \cup E^{-1}$ then gives us a partially ordered set, the *generalised poset of irreducibles (GPI)*. If C is the covering relation in the GPI, then $(J, M, E \cap C)$ is the *reduced M-representation* of L.

Tamari Lattices





Some special representations

Theorem

L a finite distributive lattice with reduced M-representation (J, M, E). Then the reduced J and M are isomorphic and E is an isomorphism between them.

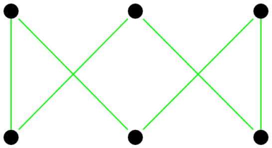
Theorem

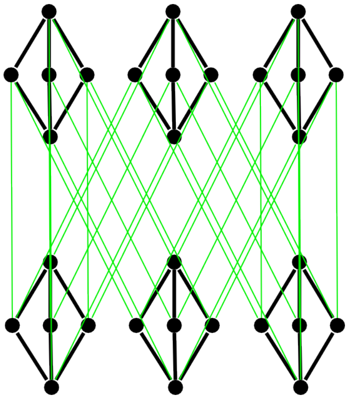
L a finite semidistributive lattice with reduced M-representation (J, M, E). Then the reduced J and M are the same size and E is a matching between them.

Theorem

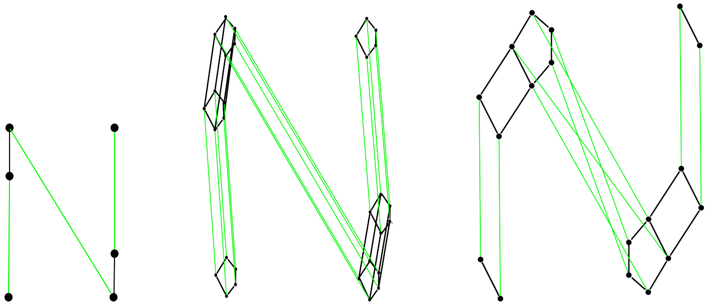
L a finite lattice with reduced M-representation (J, M, E). Then the reduced M-representation of $O(L)$ is $(J \times L^{\delta}, M \times L^{\delta}, E \times i)$.

Example: M_3





Example: Tamari Lattice T_4



Units in the seminearring

Theorem

Let L be a finite lattice. Then the order automorphisms are the lattice automorphisms

So in a tolerance simple lattice, all lattice automorphisms are polynomials.

Example: Let $a, b, c \in M_n$ be such that $a \neq b \neq c$. Then $p(x) = ((a \wedge x) \vee b) \wedge c$ maps a and 1 to c and all other elements to 0.

Let f be a permutation of the n non bound elements of M_n, g a mapping that respects $x \neq g(x) \neq f(x)$. Then the polynomial

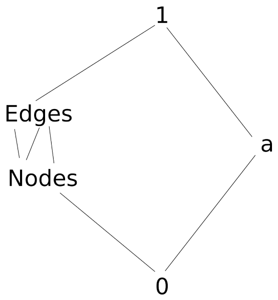
$$p_f(x) = \bigvee_a ((a \wedge x) \vee g(a)) \wedge f(a)$$

gives the polynomial function equal to f.

All finite groups

Theorem (Schweigert 75)

For all finite groups G there is a finite lattice L such that $u(Pol_1(L)) \cong G$.



Polynomial and Lattice Rank

The *rank* of a polynomial is the number of times x appears in it. The rank of a polynomial function is the lowest rank amongst all polynomials inducing that function. The rank of a lattice is the largest rank of a polynomial function on that lattice.

Theorem (Dorninger, Wiesenbauer 74)

The rank of a lattice is 1 iff that lattice is distributive

Theorem

If two polynomial units on a lattice L are conjugate by the action of $Aut(L)$, then they have the same rank.

Polynomial and Lattice Rank

Theorem

If $p = p_1 \vee p_2$ in a unit for three distinct polynomials, then p_1, p_2 are not invertible, and dually for $p = p_1 \wedge p_2$.

Theorem (Weinert)

$(S, +, *)$ a finite seminearfield. Then $(S, +)$ is a rectangular semigroup, $S = A \times B$ with $(a, b) + (c, d) = (a, d)$ and S is also a group product of the trivially intersecting subgroups A, B, $S = A * B = B * A$.

Ongoing Work

Goal: How to move from the reduced M-representation of a lattice directly to the reduced M-representation of the polynomials on that lattice.

Question: Is tolerance simplicity recognisable from the reduced M-representation?

Question: What ranks arise for polynomial units? Are nontrivial polynomial units always of maximal rank?