FINITE WEAKLY DIVISIBLE NEARRINGS

PETER MAYR AND FIORENZA MORINI

Abstract. A nearring $(N, +, *)$ is called weakly divisible iff for all elements $a, b \in N$ there exists an element $x \in N$ such that $x*a = b$ or $x*b = a$. All such finite zerosymmetric nearrings are determined.

1. Introduction

A right nearring $(N, +, *)$ is a $(2, 2)$-algebra, where $(N, +)$ is a (not necessarily abelian) group, $(N, *)$ is a semigroup and the distributive law $(a + b) * c = a * c + b * c$ holds.

Definition 1. A nearring $(N, +, *)$ is called *weakly divisible* iff

$$\forall a, b \in N \ \exists x \in N \ : \ x * a = b \text{ or } x * b = a$$

This condition holds for example for integral planar nearrings, [Cla92].

Equivalently, a nearring $(N, +, *)$ is weakly divisible (wd) if and only if the set of N-subgroups of $(N, +)$ is linearly ordered and each element of N has a left identity, that is, $n \in N * n$ for each $n \in N$, ([BP99] Proposition 5). In particular, the set of (left) ideals and the set of left annihilators of a wd nearring are linearly ordered.

For the finite zerosymmetric case the structure of wd nearrings is very similar to that of integral planar ones. In this note a characterization for all such wd nearrings is given, which resembles the construction of planar nearrings by G. Ferrero from a group $(N, +)$ and a group of fixed-point-free automorphisms of $(N, +)$ in [Fer70]. This is also a generalization of the results of two articles by A. Benini and F. Morini, [BM98a] and [BM98b], wherein wd nearrings with cyclic additive groups of prime power order are determined.

2. Construction of wd nearrings

Theorem 1. Let $(N, +)$ be a finite group and let the following hold:

(a) Let ψ be a nilpotent endomorphism of $(N, +)$ with $\operatorname{Im} \psi^{r-1} = \operatorname{Ker} \psi$, where r is minimal such that $\psi^r = 0$. (Let $\psi^0 := \mathrm{id}$.)

(b) Let Φ be a group of automorphisms on $(N, +)$ such that $\Phi\psi \subseteq \psi\Phi$ and

$$\forall i, \ 0 \leq i < r \ \forall n \in N \setminus \operatorname{Im} \psi \quad |\psi^i \Phi(n)| = |\psi^i \Phi|$$

(c) Let $E \subseteq N$ be a complete set of orbit representatives for Φ on $N \setminus \operatorname{Im} \psi$ such that

$$\forall i, \ 0 \leq i < r \ \forall e_1, e_2 \in E \quad (\ \psi^i \Phi(e_1) = \psi^i \Phi(e_2) \Rightarrow \psi^i(e_1) = \psi^i(e_2) \ )$$

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For any $x \in N, e \in E, \varphi \in \Phi$ and $0 \leq i \leq r$ define

$$x * \psi^i \varphi(e) := \psi^i \varphi(x)$$

Then $(N, +, *)$ is a zerosymmetric wd nearring.

Proof. First show that for all $n \in N$ there exists $0 \leq i \leq r$ and $\varphi \in \Phi, e \in E$ such that n has a presentation as $n = \psi^i \varphi(e)$. There exists a uniquely determined integer i such that $n \in \operatorname{Im} \psi^i \setminus \operatorname{Im} \psi^{i+1}$. Thus $n = \psi^i(m)$ for some $m \in N \setminus \operatorname{Im} \psi$ and there is $e \in E$ and $\varphi \in \Phi$ such that $m = \varphi(e)$. While there is only one choice for i with $0 \leq i \leq r$ such that $n = \psi^i \varphi(e)$ both $\varphi \in \Phi$ and $e \in E$ are not uniquely determined.

Suppose, that $\psi^i \varphi_1(e_1) = \psi^i \varphi_2(e_2)$ or, equivalently, $\varphi_1(e_1) + \operatorname{Ker} \psi^i = \varphi_2(e_2) + \operatorname{Ker} \psi^i$. By condition (a) $\operatorname{Ker} \psi^i = \operatorname{Im} \psi^{r-i}$ and $\operatorname{Im} \psi^{r-i}$ is invariant under automorphisms in Φ by (b). Thus $\Phi(e_1 + \operatorname{Ker} \psi^i) = \Phi(e_2 + \operatorname{Ker} \psi^i)$ and hence $e_1 + \operatorname{Ker} \psi^i = e_2 + \operatorname{Ker} \psi^i$ by (c). Now $\varphi_1(e_1) + \operatorname{Ker} \psi^i = \varphi_2(e_1) + \operatorname{Ker} \psi^i$ and $\psi^i \varphi_1(e_1) = \psi^i \varphi_2(e_1)$. The cardinality condition in (b) states a bijection between $\psi^i \Phi(e_1)$ and $\psi^i \Phi$. If two maps $\psi^i \varphi_1$ and $\psi^i \varphi_2$ coincide on one element $e_1 \in N \setminus \operatorname{Im} \psi$, then they coincide on N as a whole. Thus $\psi^i \varphi_1 = \psi^i \varphi_2$ and the product $x * n$ is the same for all choices of $e \in E, \varphi \in \Phi$ such that $n = \psi^i \varphi(e)$.

For associativity

$$(x * \psi^i \varphi_1(e_1)) * \psi^j \varphi_2(e_2) = x * (\psi^i \varphi_1(e_1) * \psi^j \varphi_2(e_2))$$

or, equivalently,

$$\psi^j \varphi_2 \psi^i \varphi_1(x) = x * \psi^j \varphi_2 \psi^i \varphi_1(e_1)$$

for all $x \in N, e_1, e_2 \in E, \varphi_1, \varphi_2 \in \Phi$ and integers i, j has to be shown. Equality is proved by substituting $\varphi_2 \psi^i$ with $\psi^i \varphi_2'$ for some $\varphi_2' \in \Phi$ according to condition (b).

Distributivity holds by definition of the multiplication via endomorphisms.

Let $a = \psi^i(\varphi_1(e_1))$ and $b = \psi^j(\varphi_2(e_2))$ with $0 \leq i \leq j \leq r$. Then $x = \varphi_1^{-1} \psi^{j-i} \varphi_2(e_2)$ solves the equation $x * a = b$ and $(N, +, *)$ is wd. $\square$

In the sequel a nearring defined as in Theorem 1 is denoted by $W(N, \psi, \Phi, E)$.

For the choice of $\psi = 0$ the condition (a) of the above theorem is trivially fulfilled and (b), (c) are equivalent to Φ being a group of fixed-point-free automorphisms. In this case the construction method is then equivalent to the construction method for planar nearrings according to G. Ferrero [Fer70] and $W(N, 0, \Phi, E)$ is a integral planar nearring.

The following natural questions arise and are answered in this note:

(a) For which choice of $(N, +), \psi, \Phi$ and E are the conditions (a), (b) and (c) as required in Theorem 1 fulfilled?
(b) Are all wd nearrings obtained as some $W(N, \psi, \Phi, E)$?
(c) Which "inputs" to Theorem 1 give rise to isomorphic nearrings?

3. Structure of $W(N, \psi, \Phi, E)$

Condition (a) in Theorem 1 poses a restriction on the endomorphism ψ as well as on the additive group $(N, +)$. In particular, it shows that $(N, +)$ has a normal series

$$N = \operatorname{Im} \psi^0 \rhd \operatorname{Im} \psi^1 \rhd \operatorname{Im} \psi^2 \rhd \cdots \rhd \operatorname{Im} \psi^{r-1} \rhd \operatorname{Im} \psi^r = \{0\}$$

where each factor $\operatorname{Im} \psi^i / \operatorname{Im} \psi^{i+1}$ for $0 \leq i < r$ is isomorphic to $\operatorname{Im} \psi^{r-1} = \operatorname{Ker} \psi$, thus $|N| = |\operatorname{Ker} \psi|^r$.

Moreover, $\operatorname{Im}\psi^i/\operatorname{Im}\psi^j \cong \operatorname{Ker}\psi^{j-i}$ for all $0 \le i \le j \le r$; that is, for $0 < k < r$ all subseries of k consecutive terms of the above normal series describe isomorphic groups.

The next result shows the relations between $(N,+), \psi, \Phi, E$ and the characteristics of the nearring $W(N,\psi,\Phi,E)$ constructed thereof.

Proposition 1. Let $N = W(N,\psi,\Phi,E)$ with r minimal such that $\psi^r = 0$. Then

(a) E is the set of right identities of $(N,+,*)$.
(b) The set of N-subgroups of N equals $\{\operatorname{Im}\psi^i\}_{0\le i\le r}$.
(c) The set of ideals of N equals $\{\operatorname{Im}\psi^i\}_{0\le i\le r}$.
(d) The elements of $\operatorname{Im}\psi$ are nilpotent and $\operatorname{Im}\psi$ is a prime ideal.
(e) $N/\operatorname{Im}\psi$ is either a zerosymmetric constant nearring or an integral planar nearring.

Proof. Straightforward calculations. □

In the case of Φ being a group of fixed-point-free automorphisms on $(N,+)$ the condition (c) of Theorem 1 can be exchanged by a more convenient one:

Proposition 2. Let $(N,+), \psi$ be according to the assumption (a) of Theorem 1 and let Φ be a group of fixed-point-free automorphisms on $(N,+)$ such that $\Phi\psi \subseteq \psi\Phi$. Then the second condition in (b) is fulfilled.

Let E be a complete set of orbit representatives of Φ on $N\backslash\operatorname{Im}\psi$. Then the condition (c) is fullfilled if and only if $E = \bigcup_{\hat{e}\in\hat{E}} \hat{e}$ where $\hat{E}$ is a set of all nonzero orbit representatives of Φ acting on $N/\operatorname{Im}\psi$.

Proof. The second condition on Φ in (b) is fulfilled, since Φ is fixed-point-free on all normal subgroups $\operatorname{Ker}\psi^i$ and all factors $N/\operatorname{Ker}\psi^i$ for $0 \le i < r$.

Suppose that the set of orbit representatives E fulfills condition (c). In particular for all pairs of elements $e_1, e_2 \in E$ $\Phi(e_1) + \operatorname{Ker}\psi^{r-1} = \Phi(e_2) + \operatorname{Ker}\psi^{r-1}$ implies that $e_1 + \operatorname{Ker}\psi^{r-1} = e_2 + \operatorname{Ker}\psi^{r-1}$. Equivalently, if two orbits $\Phi(e_1), \Phi(e_2)$ are congruent modulo $\operatorname{Im}\psi$, then their respective representatives e_1, e_2 also have to be congruent modulo $\operatorname{Im}\psi$. If $\hat{E}$ is a set of all nonzero orbit representatives of Φ acting on $N/\operatorname{Im}\psi$, then in any case E has to be a subset of the union of cosets $\hat{e} = e + \operatorname{Im}\psi$ with $\hat{e} \in \hat{E}$.

Since Φ is fixed-point-free on N and in particular on $N/\operatorname{Im}\psi$, the size of orbits of Φ on $N \setminus \operatorname{Im}\psi$ equals the size of nonzero orbits of Φ acting on $N/\operatorname{Im}\psi$. The number of orbits on $N \setminus \operatorname{Im}\psi$ is $(|N| - |\operatorname{Im}\psi|)/|\Phi|$ and equals $(|N/\operatorname{Im}\psi| - 1) * |\operatorname{Im}\psi|/|\Phi|$ the number of elements in $\bigcup_{\hat{e}\in\hat{E}} \hat{e}$. Thus E and $\bigcup_{\hat{e}\in\hat{E}} \hat{e}$ coincide.

For the converse the observation that $E = \bigcup_{\hat{e}\in\hat{E}} \hat{e}$ is indeed a set of orbit representatives of Φ on $N/\operatorname{Im}\psi$ if Φ is fixed-point-free suffices again. Then the condition (c) is clearly fulfilled for E. □

4. All wd nearrings

All finite zerosymmetric wd nearrings are obtained by the construction method of Theorem 1.

Theorem 2. Let $(N,+,*)$ be a finite zerosymmetric wd nearring with Q the maximal proper N-subgroup of $(N,+)$. Then the following hold:

(a) There is an element $q \in Q$ such that $Q = N * q$ and the mapping $\psi : x \mapsto x * q$ is a nilpotent endomorphism on $(N,+)$.

(b) $\Phi := \{\varphi_c : x \mapsto x * c \mid c \in N \setminus Q\}$ is a group of automorphisms on $(N, +)$.
(c) The set of right identities E of $(N, +, *)$ is a set of orbit representatives of $N \setminus Q$ under Φ.

ψ, Φ, E fulfill the conditions (a), (b), (c) in Theorem 1 and $(N, +, *) = W(N, \psi, \Phi, E)$.

Proof. (a) The family of N-subgroups $\{N * p \mid p \in Q\}$ of $(N, +)$ is linearly ordered. Let $N * q$ for some q be the unique maximal element therein. Suppose that $Q \neq N * q$ and let $q' \in Q \setminus N * q$. Then $q \in N * q'$ and $N * q \subseteq N * q'$ implying that $N * q = N * q'$, since $N * q$ is maximal. Now, $q' \in N * q$ in contradiction to the assumption. Thus $Q = N * q$.

Consider the chain of N-subgroups $N \geq N * q \geq N * q^2 \geq \dots$. There is an integer r such that $N * q^{r-1} > N * q^r = N * q^{r+1}$. Suppose that $\operatorname{Ker}\psi < N * q^r$. Then $(N * q^r)/\operatorname{Ker}\psi \cong N * q^{r+1}$ implies $\operatorname{Ker}\psi = \{0\}$ and $N = Q$ in contradiction to our assumption.

Thus $\operatorname{Ker}\psi \geq N * q^r$ and ψ is nilpotent by $\{0\} = N * q^{r+1} = N * q^r$.

(b) Let $c \notin N * q$. Then $N * c = N$ and $\varphi_c : x \mapsto x * c$ is a group-automorphism of $(N, +)$. Since $N \setminus N * q$ is closed under multiplication, the set of automorphisms $\{\varphi_c : x \mapsto x * c \mid c \in N \setminus Q\}$ forms a group.

(c) By (b) the solution of $x * c = c$ for each $c \in N \setminus Q$ is unique. Let it be denoted by e_c. Then $c = \varphi_c(e_c)$ is an element of the orbit $\Phi(e_c)$ and by $\varphi_c(x * e_c) = \varphi_c(x)$ also $x * e_c = x$ holds for all $x \in N$. Therefore e_c is a right identity of $(N, +, *)$ and this right identity is unique in the orbit $\Phi(c)$.

Thus the set of right identities are orbit representatives as required.

Let $n \in N$ be arbitrary. Then n has a representation as $n = c * q^i$ with $c \in N \setminus N * q$ and some integer $i \geq 0$. For any $x \in N$ the product $x * n$ equals $x * c * q^i = \psi^i(x * c) = \psi^i \varphi_c(x)$ where $\varphi_c(e_c) = c$.

What remains to be shown is that the additional conditions on ψ, Φ, E as stated in the assumptions (a), (b), (c) of Theorem 1 hold.

$\operatorname{Im}\psi^{r-1} \subseteq \operatorname{Ker}\psi$, since $\psi^r = 0$. To prove the converse inclusion, let $k \in \operatorname{Ker}\psi$ be represented as $k = \psi^i \varphi(e)$ for some integer j and $e \in E, \varphi \in \Phi$. By definition $0 = \psi(k) = \psi^{i+1}\varphi(e)$ and hence $x * 0 = x * \psi^{i+1}\varphi(e) = \psi^{i+1}\varphi(x)$ for all elements $x \in N$. This implies that $N * 0 = \psi^{i+1}(N)$ and finally $\{0\} = \operatorname{Im}\psi^r = \operatorname{Im}\psi^{i+1}$. Thus $i \geq r - 1$ and $k \in \operatorname{Im}\psi^{r-1}$.

Associativity implies that $\Phi\psi \subseteq \psi\Phi$: Let $e \in E$ and $\varphi \in \Phi$ be arbitrary but fixed. Then $(x * \psi(e)) * \varphi(e) = x * (\psi(e) * \varphi(e))$ for all $x \in N$. There exist $e' \in E$ and $\varphi' \in \Phi$ such that $\varphi\psi(e) = \psi^i \varphi'(e')$ for some integer i. This substition yields $\varphi\psi(x) = \psi^i \varphi'(x)$ and subsequently $\varphi \operatorname{Im}\psi = \operatorname{Im}\psi^i$. Thus $i = 1$ and $\varphi\psi = \psi\varphi'$.

Let $\psi^i \varphi_1(e_1) = \psi^i \varphi_2(e_2)$ for $e_1, e_2 \in E, \varphi_1, \varphi_2 \in \Phi$ and $0 \leq i < r$. The equality $x * \psi^i \varphi_1(e_1) = x * \psi^i \varphi_2(e_2)$ implies that $\psi^i \varphi_1(x) = \psi^i \varphi_2(x)$ for all $x \in N$.

With the choice of $e_1 = e_2$ this gives the bijection between $\psi^i \Phi(e_1)$ and $\psi^i \Phi$ as stated in condition (b) of Theorem 1.

On the other hand, $\psi^i \varphi_1(e_1) = \psi^i(e_2)$ implies that $\psi^i \varphi_1 = \psi^i$ and hence $\psi^i(e_1) = \psi^i(e_2)$, as demanded by condition (c) of Theorem 1. □

5. Isomorphisms of wd nearrings

Theorem 3. The nearrings $W(N_1, \psi_1, \Phi_1, E_1)$ and $W(N_2, \psi_2, \Phi_2, E_2)$ are isomorphic if and only if the following conditions are satisfied:

(a) There exists a group isomorphism α from $(N_1, +_1)$ to $(N_2, +_2)$.
(b) $E_2 = \alpha(E_1)$.
(c) $\Phi_2 = \alpha\Phi_1\alpha^{-1}$.
(d) $\psi_2 \in \alpha\psi_1\Phi_1\alpha^{-1}$.

Proof. Let the multiplication in $W_1 := W(N_1, \psi_1, \Phi_1, E_1)$ be denoted with $*_1$ and the multiplication in $W_2 := W(N_2, \psi_2, \Phi_2, E_2)$ with $*_2$.

"$\Rightarrow$": Suppose that α is a nearring isomorphism from W_1 to W_2. Then in particular α is a group isomorphism from $(N_1, +)$ to $(N_2, +)$. Necessarily, α maps the set of right identities of W_1 to right identities of W_2, thus $\alpha(E_1) = E_2$. Furthermore, nilpotency is invariant under homomorphisms, hence $\alpha(\operatorname{Im}\psi_1) = \operatorname{Im}\psi_2$.

For all $\varphi_1 \in \Phi_1$ and $e_1 \in E_1$ the equation $\alpha(x *_1 \varphi_1(e_1)) = \alpha(x) *_2 \alpha\varphi_1(e_1)$ holds and is equivalent to $\alpha\varphi_1(x) = \alpha(x) *_2 \alpha\varphi_1(e_1)$. Now, $\alpha\varphi_1(e_1) \notin \operatorname{Im}\psi_2$ for cardinality reasons. Thus $\alpha\varphi_1(e_1) = \varphi_2(e_2)$ for some $\varphi_2 \in \Phi_2, e_2 \in E_2$ and $\alpha\varphi_1(x) = \varphi_2\alpha(x)$ for all $x \in N$, finally $\Phi_2 \supseteq \alpha\Phi_1\alpha^{-1}$ and equality follows again from a cardinality argument.

Consider $\alpha(x *_1 \psi_1(e_1)) = \alpha(x) *_2 \alpha\psi_1(e_1)$ for $e_1 \in E_1$. Since $\alpha\psi_1(e_1) = \psi_2\varphi_2(e_2)$ for some $\varphi_2 \in \Phi_2, e_2 \in E_2$, this can be rewritten as $\alpha\psi_1(x) = \psi_2\varphi_2\alpha(x)$ and $\psi_2 \in \alpha\psi_1\Phi_1\alpha^{-1}$.

"$\Leftarrow$": It suffices to show that for all $x \in N_1$ and $\varphi \in \Phi_1, e \in E_1, 0 \leq i \leq r$

$$\alpha(x *_1 \psi_1^i\varphi(e)) = \alpha(x) *_2 \alpha\psi_1^i\varphi(e)$$

Let $\psi_1 = \alpha^{-1}\psi_2\varphi_2\alpha$ with $\varphi_2 \in \Phi_2$:

$$\begin{aligned} \alpha(x) *_2 \alpha\psi_1^i\varphi(e) &= \alpha(x) *_2 \alpha(\alpha^{-1}\psi_2\varphi_2\alpha)^i\varphi(e) \\ &= \alpha(x) *_2 (\psi_2\varphi_2)^i\alpha\varphi(e) \\ &= \alpha(x) *_2 (\psi_2\varphi_2)^i(\alpha\varphi\alpha^{-1})\alpha(e) \end{aligned}$$

Since $\alpha\varphi\alpha^{-1} \in \Phi_2$ and $\alpha(e) \in E_2$, this yields

$$\begin{aligned} \alpha(x) *_2 \alpha\psi_1^i\varphi(e) &= (\psi_2\varphi_2)^i(\alpha\varphi\alpha^{-1})\alpha(x) \\ &= (\alpha\psi_1\alpha^{-1})^i(\alpha\varphi\alpha^{-1})\alpha(x) \\ &= \alpha\psi_1^i\varphi(x) \\ &= \alpha(x *_1 \psi_1^i\varphi(e)) \end{aligned}$$

which completes the proof. □

6. An example: wd nearrings on cyclic groups

For the choice of $(N, +)$ being a cyclic group, the initial conditions on the nilpotent endomorphism ψ, automorphism group Φ and representatives E can be restated in a more convenient way.

Proposition 3. Let ψ be a nilpotent endomorphism on $(\mathbb{Z}_n, +)$ with $|\operatorname{Ker}\psi| = q$ not prime and let Φ be a subgroup of $\operatorname{Aut}(\mathbb{Z}_n, +)$ with E a set of orbit representatives for $\mathbb{Z}_n \setminus \operatorname{Im}\psi$ under Φ.

Then the conditions (a), (b), (c) in Theorem 1 are fulfilled iff the following conditions hold:

(a) There exists an integer r such that $n = q^r$.
(b) Φ is a group of fixed point free automorphisms on $\mathbb{Z}_{q^r}$.
(c) $E = \bigcup_{\hat{e} \in \hat{E}} \hat{e}$ with $\hat{E}$ a set of orbit representatives for Φ acting on $\mathbb{Z}_{q^r}/q^{r-1}\mathbb{Z}_{q^r}$.

Proof. "$\Rightarrow$": Since the size n of the additive group is a power of $|\operatorname{Ker}\psi| = q$ by the first paragraph of Section 3, the assertions on n and ψ hold.

The set of fixed points of an automorphism $\varphi \in \Phi$ forms an N-subgroup of $\mathbb{Z}_{q^r}$ and therefore equals $q^s\mathbb{Z}_{q^r}$ with $0 \leq s \leq r$ by Proposition 1 (b).

Suppose that there exists an automorphism $\varphi \in \Phi$ with $\varphi(x) = fx$ for some integer f and a non-trivial set of fixed points $q^s\mathbb{Z}_{q^r}$ with $0 < s < r$. Thus $\gcd(f-1, q^r) = q^{r-s}$.

Now, for q not prime let p be a either an odd prime divisor of q or $p = 2$ if q is a power of 2 and consider $\varphi^p(x) = f^p x$. A number theoretical consideration yields a contradiction:

$$\begin{aligned} \frac{f^p - 1}{f - 1} &= f^{p-1} + \cdots + f + 1 \\ &\equiv \underbrace{1 + \cdots + 1}_{p \text{ times}} \mod q \\ &\equiv p \mod q \end{aligned}$$

Since $\gcd(f^p - 1, q^r) = pq^{r-s}$, the set of fixed points of $\varphi^p \in \Phi$ violates the structure of N-groups as stated above.

Since Φ is fixed-point-free, E is determined by Proposition 2.

"$\Leftarrow$": Proposition 2. $\square$

Theorem 2 in [BM98b] deals with the case that the size of $\operatorname{Ker}\psi$ is a prime. Then the automorphisms of Φ need not be fixed point free, but still a corresponding condition on the orbit representatives can be given, as is shown there.

Functions for the construction of wd nearrings and geometries derived from them are available as part of SONATA [Tea00].

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Peter Mayr, Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria

E-mail address: peter.mayr@algebra.uni-linz.ac.at

Fiorenza Morini, Dipartimento di Matematica - Facolta' di Ingegneria - Universita' degli Studi di Brescia, via Valotti 9, 25133 Brescia, Italia

E-mail address: morini@bsing.ing.unibs.it