# A topological property of polynomial functions on $\mathrm{GL}(2, \mathbb{R})$

Günter Landsmann, Peter Mayr, and Josef Schicho

April 12, 2006

**Abstract**

A function from the group $\mathrm{GL}(2, \mathbb{R})$ to itself is called polynomial if it can be written as some product of constant functions, the identity function, and the function that maps every element to its inverse. We give a necessary topological condition for a function to be polynomial. As a consequence we prove that transposition is not polynomial.

**Mathematics Subject Classification 2000.** 08A40 (20G20)
**Keywords.** Polynomial functions, linear groups, homotopy class groups.

## Introduction

Polynomial functions (see Section 1 for definitions) have been studied for several classes of finite groups. An overview of existing results and references can be found in [2]. From the paper [1] we obtain a characterization of the so-called locally polynomial functions on the group $\mathrm{GL}(n, K)$. These are the functions that can be interpolated by a polynomial function on any given finite set of points. By the algebraic criteria given in [1] a function $f : \mathrm{GL}(n, K) \to \mathrm{SL}(n, K)$ is locally polynomial if and only if

$$f(kx) = f(x) \text{ for all } x \in \mathrm{GL}(n, K) \text{ and for all } k \in K^*. \tag{1}$$

Consequently transposition on any general linear group over any field turns out to be a locally polynomial function, and transposition is polynomial on any general linear group over any finite field.

The second author raised the question whether transposition on infinite general linear groups is polynomial in an interdisciplinary seminar, where the third author proposed to bring in topological ideas. Finally we could answer the question negatively for the case $\mathrm{GL}(2, \mathbb{R})$ by a method involving topology and algebra. It should be noted that the topological arguments even resisted subsequent trials to eliminate them. We still have no idea for a purely algebraic proof.

The structure of the paper is the following. In Section 2 we give the structure of the homotopy class group of $\mathrm{GL}(2, \mathbb{R})$. In our main result Theorem 2 we determine the polynomial functions modulo homotopy. From this description we then obtain that transposition is not equivalent to a polynomial function modulo homotopy. Hence transposition is not polynomial on $\mathrm{GL}(2,\mathbb{R})$. The negative answer for $\mathbb{R}$ also implies the negative answer for all subfields of $\mathbb{R}$. More generally, the negative result can be transferred to arbitrary ordered fields (see Section 4).

# 1 Notation

Let $(G,\cdot)$ be a group. A *unary polynomial function* $p : G \to G$ is a function that can be written in the form

$$p(x) := a_1 x^{e_1} a_2 x^{e_2} \cdots a_n x^{e_n} a_{n+1},$$

where $n \in \mathbb{N}$, $a_1, \ldots, a_{n+1}$ are in $G$, and $e_1, \ldots, e_n$ are integers (see [5], [6, Definition 4.4]). The set of all unary polynomial functions on $G$ will be denoted by $\mathrm{P}(G)$, the set of all functions from $G$ into $G$ by $G^G$. For $f, g \in G^G$ we define the product $fg \in G^G$ by $fg\,(x) = f(x) \cdot g(x)$ for all $x \in G$. By this multiplication of functions, $(G^G, \cdot)$ is a group, and $(\mathrm{P}(G), \cdot)$ is the subgroup of $(G^G, \cdot)$ that is generated by the identity function and the constant functions on $G$. Inner automorphisms of the group $G$ are particular examples of polynomial functions. We will denote the multiplicative inverse of $f \in G^G$ by $f^{-1}$, which is not to be mistaken for the inverse function with respect to composition. For the usual composition of functions $f, g$ with appropriate source and range we will write $f \circ g$.

Now assume that $G$ is a topological group. Recall that two continuous functions $f, g : G \to G$ are *homotopic* if there exists a continuous function $h : G \times [0,1] \to G$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in G$. For any continuous function $f$ on $G$ we let $[f]$ denote the homotopy class of $f$. We write $[G,G] := \{[f] \mid f \text{ is continuous on } G\}$. Since the multiplication of functions is well-defined modulo homotopy, we have a group $([G,G], \cdot)$. Polynomial functions are continuous. Hence their classes form a subgroup. This group of polynomial functions modulo homotopy is generated by the class of the identity and the classes of all constant functions (it suffices to choose one constant from each connected component of $G$).

# 2 Computation of Homotopy Class Groups

We recall that two spaces $X, Y$ are *homotopy equivalent* if there are two continuous maps $u : X \to Y$ and $v : Y \to X$ such that the composed function $u \circ v$ is homotopic to $\mathrm{id}_Y$ and $v \circ u$ is homotopic to $\mathrm{id}_X$. If $X$ and $Y$ are homotopy equivalent, then

$$\alpha : [X,X] \to [Y,Y],\ [f] \mapsto [u \circ f \circ v],$$

is well-defined and an isomorphism.

For the rest of the paper we will use the following notation. We write $G := \mathrm{GL}(2,\mathbb{R})$, $O := \mathrm{O}(2,\mathbb{R})$, $O_+ := \{x \in O \mid \det x = 1\}$, $O_- := \{x \in O \mid \det x = -1\}$, and $S^1 := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

It is well-known that the topological groups $G$ and $O$ are homotopy equivalent by the following argument (cf. [4], p. 293). The Gram-Schmidt orthogonalization process applied to the rows of matrices in $G$ provides a continuous function

$$u : G \to O,\ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1/|a_1| \\ (a_2 - \frac{a_2\cdot a_1}{a_1\cdot a_1}a_1)/|a_2 - \frac{a_2\cdot a_1}{a_1\cdot a_1}a_1| \end{pmatrix}. \tag{2}$$

Here $x \cdot y$ denotes the standard scalar product of row vectors $x, y \in \mathbb{R}^2$ and $|x| := \sqrt{x \cdot x}$. For $v := \mathrm{id}_O$ we then have that $u \circ v = \mathrm{id}_O$ and that $v \circ u = u$ is homotopic to $\mathrm{id}_G$. Thus $G$ and $O$ are homotopy equivalent.

By the bijections

$$\begin{aligned} o_+ &: S^1 \mapsto O_+,\ (\cos\varphi, \sin\varphi) \mapsto \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \\ o_- &: S^1 \mapsto O_-,\ (\cos\varphi, \sin\varphi) \mapsto \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}, \end{aligned}$$

both $O_+$ and $O_-$ are homotopy equivalent to $S^1$.

For $a \in O$ we define $c_a : O \to O, x \mapsto a$. For $f : O_+ \to O$ and $g : O_- \to O$ we define the function $f \uplus g : O \to O$ by $(f \uplus g)|_{O_+} = f$ and $(f \uplus g)|_{O_-} = g$. We denote $\mathrm{diag}(1,-1)$ by $r$ and $\mathrm{diag}(1,1)$ by $1$. Let

$$\begin{aligned} i_+ &:= \mathrm{id}_{O_+} \uplus c_1|_{O_-}, \\ s_+ &:= c_r|_{O_+} \uplus c_1|_{O_-}, \\ i_- &:= c_1|_{O_+} \uplus \mathrm{id}_{O_-}, \\ s_- &:= c_1|_{O_+} \uplus c_r|_{O_-}. \end{aligned}$$

Then $i_+ i_- = \mathrm{id}_O$ and $s_+ s_- = c_r$.

We are now able to give an explicit description of the homotopy class group $[O,O]$ and hence of the isomorphic group $[G,G]$.

**Theorem 1.** *Let $A$ be the subgroup of $(O^O, \cdot)$ that is generated by $i_+$ and $s_+$, and let $B$ be the subgroup of $(O^O, \cdot)$ that is generated by $s_- i_-$ and $s_-$.*

1. *Then $i_+$ generates a subgroup of index 2 in $A$, and $s_- i_-$ generates a subgroup of index 2 in $B$. The groups $A$ and $B$ are isomorphic to the infinite dihedral group. Further $AB$ is a group, and the product $AB$ is direct.*

2. *For each continuous function $f$ on $O$ there exist uniquely determined $a, b \in \{0,1\}$ and $c, d \in \mathbb{Z}$ such that $f$ is homotopic to ${s_+}^a {i_+}^c {s_-}^b (s_- i_-)^d$.*

3. *The group $[O,O]$ is isomorphic to $AB$.*

*Proof.* First we show (1). We note that $c_1$ is the identity element of the group $(O^O, \cdot)$. From the definition of the functions it is straightforward that the multiplicative order of $i_+$ is infinite, that $s_+$ has order 2, and that $s_+ i_+ s_+ = {i_+}^{-1}$.

Hence $i_+, s_+$ generate a copy of the infinite dihedral group. Similarly $i_-$ and $s_-$ have order 2, $s_-i_-$ has infinite order, and $s_-(s_-i_-)s_- = (s_-i_-)^{-1}$. Hence $\langle s_-i_-, s_-\rangle$ is isomorphic to the infinite dihedral group. Since $i_+, s_+$ commute with $i_-, s_-$, we have that $AB$ is a group. As $A \cap B = \{c_1\}$, the product $AB$ is direct.

Next we prove (2). Let $f : O \to O$ be a continuous function. For $f_+ := f|_{O_+}$ and $f_- := f|_{O_-}$ we have $f = f_+ \uplus f_-$. Since $f_+$ is continuous, its image is either contained in $O_+$ or in $O_-$. In the latter case the product function $(c_r|_{O_+})f_+$ maps $x$ to the element $rf_+(x)$, which is in $O_+$ for all $x \in O_+$. Hence we have $a \in \{0, 1\}$ such that $(c_r|_{O_+})^a f_+(O_+) \subseteq O_+$. Let ${o_+}^{-1}$ denote the inverse function of $o_+$ with respect to composition. Then

$$\beta : [S^1, S^1] \to [O_+, O_+],\ [g] \mapsto [o_+ \circ g \circ {o_+}^{-1}]$$

is well-defined and an isomorphism. The group $[S^1, S^1]$ is an infinite cyclic group that is generated by the class of $\mathrm{id}_{S^1}$. Hence $[O_+, O_+]$ is infinite cyclic and generated by $\beta([\mathrm{id}_{S^1}]) = [\mathrm{id}_{O_+}]$. In particular $[(c_r|_{O_+})^a f_+]$ is in the cyclic group $\langle[\mathrm{id}_{O_+}]\rangle$. Thus there exists a uniquely determined integer $c$ such that $(c_r|_{O_+})^a f_+$ is homotopic to $(\mathrm{id}_{O_+})^c$.

Similarly we have $b \in \{0, 1\}$ such that $(c_r|_{O_-})^b f_-(O_-) \subseteq O_+$. With ${o_-}^{-1}$ denoting the inverse function of $o_-$ with respect to composition, we have a group isomorphism

$$\gamma : [S^1, S^1] \to [O_-, O_+],\ [g] \mapsto [o_+ \circ g \circ {o_-}^{-1}].$$

Then $\gamma([\mathrm{id}_{S^1}]) = [c_r|_{O_-}\, \mathrm{id}_{O_-}]$ since $o_+(\mathrm{id}_{S^1}({o_-}^{-1}(x))) = rx$ for all $x \in O_-$. Hence $[c_r|_{O_-}\, \mathrm{id}_{O_-}]$ generates the group $[O_-, O_+]$. There exists a uniquely determined integer $d$ such that $(c_r|_{O_-})^b f_-$ is homotopic to $(c_r|_{O_-}\, \mathrm{id}_{O_-})^d$. Hence $f_+ \uplus f_-$ is homotopic to $(c_r|_{O_+})^a(\mathrm{id}_{O_+})^c \uplus (c_r|_{O_-})^b(c_r|_{O_-}\, \mathrm{id}_{O_-})^d$, which is equal to ${s_+}^a {i_+}^c {s_-}^b (s_-i_-)^d$. Thus (2) is proved. Now (3) follows from (1) and (2). □

We note that, by Theorem 1, the group of continuous functions on $O$ actually splits into a semidirect product of $AB$ and the normal subgroup of functions that are homotopic to $c_1$. The group $[O, O]$ is isomorphic to a semidirect product $\mathbb{Z}^2 \rtimes V_4$ where $V_4$ is the Klein four-group.

# 3 Homotopy Classes of Polynomial Functions

Next we compute the subgroup of polynomial classes.

**Theorem 2.** *Let $S := \mathrm{SL}(2, \mathbb{R})$, let $\mathcal{N} := \{[f] \mid f \in \mathrm{P}(G)$ and $f(G) \subseteq S\}$, and let $i := i_+i_-$, $s := s_+s_-$. For $u$ as in (2) let*

$$\alpha : [G, G] \to [O, O],\ [f] \mapsto [u \circ f|_O].$$

*Then the group $\alpha(\mathcal{N})$ is a direct product of the group generated by $[i^4]$ and the group generated by $[i^2(si)^2]$.*

We recall that, by the definitions that precede Theorem 1, the product $i = i_+i_-$ is the identity map $\mathrm{id}_O$ and that $s = s_+s_-$ is the constant function $c_r$ for $r = \mathrm{diag}(1, -1)$ on $O$. Further $\alpha$ is an isomorphism between $[G, G]$ and $[O, O]$.

*Proof.* Let $N$ be the subgroup of $(G^G, \cdot)$ that is generated by all functions $x \mapsto [a, x^l]$ for $l \in \mathbb{Z}, a \in G$, and $x \mapsto b$ for $b \in S$. First we show that

$$N = \{f \in \mathrm{P}(G) \mid f(G) \subseteq S\}. \tag{3}$$

The inclusion $\subseteq$ is obvious. For the converse, we let $f \in \mathrm{P}(G)$ with $f(G) \subseteq S$. We assume that

$$f(x) = a_1x^{e_1}a_2x^{e_2} \cdots a_nx^{e_n}a_{n+1},$$

where $n \in \mathbb{N}$, $a_1, \dots, a_{n+1} \in G$, and $e_1, \dots, e_n \in \mathbb{Z}$. We use induction on $n$ to show that $f$ can be written as product of commutators and constants. For $n = 1$ the assumption $\det f(x) = 1$ yields $e_1 = 0$ and $f(x) = a_1a_2 \in S$ for all $x \in G$. Hence $f \in N$. Next we assume that $n \geq 2$. Then we find

$$f(x) = (\prod_{i=1}^{n-2} a_ix^{e_i}) \cdot a_{n-1}x^{e_{n-1}+e_n}a_na_{n+1} \cdot [a_na_{n+1}, x^{e_n}] \cdot [a_{n+1}, x^{e_n}]^{-1}.$$

By the induction assumption $g : G \to G, x \mapsto (\prod_{i=1}^{n-2} a_ix^{e_i})a_{n-1}x^{e_{n-1}+e_n}a_na_{n+1}$, is in $N$. Hence we obtain $f \in N$ and (3) is proved.

On $G$ all constant functions $x \mapsto a$ for $a \in G$ with $\det a > 0$ are homotopic to $x \mapsto 1$. Here 1 denotes the identity matrix in $G$. The functions $x \mapsto a$ for $a \in G$ with $\det a < 0$ are all homotopic to $x \mapsto \mathrm{diag}(1, -1)$. Further $\alpha([\mathrm{id}_G]) = [i]$ by the definition of $\alpha$. Hence the group homomorphism $\alpha$ maps the homotopy class of $x \mapsto [a, x^l]$ to $[[c_1], [i]^l] = [c_1]$ if $\det a > 0$ and to $[[s], [i]^l]$ otherwise. By (3) we then have

$$\alpha(\mathcal{N}) = \langle \{[[s], [i]^l] \mid l \in \mathbb{Z}\} \rangle.$$

We proceed to show that $\alpha(\mathcal{N})$ is generated by the classes of $i^4$ and of $i^2(si)^2$. We claim that

$$\begin{array}{lcl} [s, i^{2k}] & = & {i_+}^{4k}, \\ [s, i^{2k+1}] & = & {i_+}^{4k+2}(s_-i_-)^2 \end{array} \tag{4}$$

for all $k \in \mathbb{Z}$. Let $l \in \mathbb{Z}$. We have $[s, i^l] = [s_+, {i_+}^l]\,[s_-, {i_-}^l]$ by Theorem 1. Since ${i_-}^2 = c_1$, we note that $[s_-, {i_-}^l] = c_1$ for $l$ even and $[s_-, {i_-}^l] = (s_-i_-)^2$ for $l$ odd. Together with $[s_+, {i_+}^l] = {s_+}^{-1}{i_+}^{-l}s_+{i_+}^l = {i_+}^{2l}$, this yields (4). Hence $\alpha(\mathcal{N}) \subseteq \langle [{i_+}^4], [{i_+}^2(s_-i_-)^2] \rangle$. The converse inclusion follows trivially. By definition ${i_+}^4$ and ${i_+}^2(s_-i_-)^2$ commute. Hence $\alpha(\mathcal{N})$ is abelian. By Theorem 1 we have $\langle {i_+}^4 \rangle \cap \langle {i_+}^2(s_-i_-)^2 \rangle = \{c_1\}$. Since ${i_+}^2 = i^2$ and $(s_-i_-)^2 = (si)^2$, we finally obtain that $\alpha(\mathcal{N})$ is the direct product of $\langle [i^4] \rangle$ and $\langle [i^2(si)^2] \rangle$. The theorem is proved. $\square$

Theorem 2 yields a necessary condition for a function to be polynomial. We give two consequences for concrete functions.

**Corollary 3.** *Transposition on $G$ is not a polynomial function.*

*Proof.* We consider the function $f : G \to G, x \mapsto x^{-1}x^t$. We note that $f$ is continuous and $f(G) \subseteq \mathrm{SL}(2, \mathbb{R})$. Since $\alpha([f]) = [i^{-1}]\,[i^{-1}] = [i]^{-2}$, we obtain that $f$ is not polynomial by Theorem 2. Hence $x \mapsto x^t$ is not polynomial. □

**Corollary 4.** *Let* $1$ *denote the identity matrix in* $G$*. Then* $d : G \to G$, $x \mapsto \det(x)1$*, is not polynomial.*

*Proof.* This follows from Corollary 3 and the fact that

$$x^t = \det(x) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} \cdot x^{-1} \cdot \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

for all $x \in G$. □

From (1) we obtain that both transposition and the determinant function $d$ can be interpolated by some polynomial function on any finite set of points. We also note that we cannot decide whether the function $G \to G$, $x \mapsto (x^2)^t$, is polynomial by Theorem 2.

# 4 Consequences for Other Fields

The field $\mathbb{R}$ is not easy to describe within the algebraic language without topological concepts. Therefore one could say that the results in the previous section are not entirely algebraic results. But here is the proof that the results are also true for arbitrary ordered fields (for instance the field of rational numbers).

**Theorem 5.** *Let* $K$ *be an ordered field. Then transposition is not a polynomial function on the group* $\mathrm{GL}(2, K)$*.*

*Proof.* It is well-known that any ordered field can be embedded in a real closed field, its real closure (see [3]). Let $R$ be the real closure of $K$. We claim that $K^4$ is dense in $R^4$ with respect to the Zariski topology. It suffices to show that the complement of an arbitrary hypersurface $\{(r_1, \ldots, r_4) \in R^4 \mid f(r_1, \ldots, r_4) = 0\}$, where $f$ is a nonconstant polynomial in $R[x_1, \ldots, x_4]$, contains some point from $K^4$. If not, then the polynomial $f$ would vanish on $K^4$. Since $K$ is an infinite set, we would have the contradiction that $f = 0$. It follows that the set of regular matrices $\mathrm{GL}(2, K)$ is Zariski dense in $\mathrm{GL}(2, R)$.

Assume, indirectly, that we have integers $n, e_1, \ldots, e_n$ such that the first order formula

$$\exists (a_1, \ldots, a_{n+1}) \forall x : a_1 x^{e_1} a_2 x^{e_2} \cdots a_n x^{e_n} a_{n+1} = x^t$$

is true over the domain $\mathrm{GL}(2, K)$. Because $\mathrm{GL}(2, K)$ is Zariski dense in $\mathrm{GL}(2, R)$, the formula is also true over the domain $\mathrm{GL}(2, R)$. Because $R$ is elementary equivalent to $\mathbb{R}$, it follows that the same formula is also true over the domain $\mathrm{GL}(2, \mathbb{R})$. This contradicts Corollary 3. □

## Acknowledgments

This work was supported by the Austrian Science Foundation; the first author by project DET, Nr. P16357-N04, and by the special research area SFB 013, subproject 04, the second by grant P15691, the third by the special research area SFB 013, subproject 03.

## References

[1] E. Aichinger and P. Mayr. Polynomial functions and endomorphism near-rings on certain linear groups. *Comm. Algebra*, 31(11):5627–5651, 2003.

[2] E. Aichinger and G. Pilz. A survey on polynomials and polynomial and compatible functions. In *Proceedings of the Third International Algebra Conference (Tainan, 2002)*, pages 1–16, Dordrecht, 2003. Kluwer Acad. Publ.

[3] J. Bochnak, M. Coste, and M.-F. Roy. *Géométrie algébrique réelle*, volume 12 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[4] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[5] H. Lausch and W. Nöbauer. *Algebra of polynomials.* North-Holland, Amsterdam, London; American Elsevier Publishing Company, New York, 1973.

[6] R. N. McKenzie, G. F. McNulty, and W. F. Taylor. *Algebras, lattices, varieties, Volume I.* Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California, 1987.

Günter Landsmann
RISC
Universität Linz
4040 Linz
Austria
E-mail: `landsmann@risc.uni-linz.ac.at`

Peter Mayr
Institut für Algebra
Universität Linz
4040 Linz
Austria
E-mail: `peter.mayr@jku.at`

Josef Schicho
Johann Radon Institute
Austrian Academy of Sciences
4040 Linz
Austria
E-mail: `josef.schicho@oeaw.ac.at`