FROBENIUS COMPLEMENTS OF EXPONENT DIVIDING $2^m \cdot 9$

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ABSTRACT. We show that every group of exponent $2^m \cdot 3^n$ ($m, n \in \mathbb{N}$, $n \leq 2$) that acts freely on some abelian group is finite.

1. Results

Let V be a group, and let G be a group of automorphisms of V. We say that G acts *freely* on V if $v^g \neq v$ for all $v \in V \setminus \{1\}$ and $g \in G \setminus \{1\}$. In the literature this concept is also often called regular or fixed-point-free action of G on V.

We consider free actions of groups of finite exponent. In [1] the first author proved that groups of exponent 5 that act freely on abelian groups are finite. In the present note we show the following.

Theorem 1.1. *Let V be an abelian group, and let G be a group of automorphisms of V. If G has exponent $2^m \cdot 3^n$ for $0 \leq m$ and $0 \leq n \leq 2$ and G acts freely on V, then G is finite.*

Every finite group that acts freely on an abelian group is isomorphic to a Frobenius complement in some finite Frobenius group (see Lemma 2.6). Let G be as in Theorem 1.1. By the classification of finite Frobenius complements (see [6]) the factor of G by its maximal normal 3-subgroup is isomorphic to a cyclic 2-group, a generalized quaternion group, $\mathrm{SL}(2,3)$, or the binary octahedral group of size 48.

Corollary 1.2. *Let F be a near-field whose multiplicative group has exponent $2^m \cdot 3^n$ for $0 \leq m$ and $0 \leq n \leq 2$. Then either $|F| \in \{2^2, 3^2, 5^2, 7^2, 17^2\}$ or F is a finite field of prime order.*

We note that there exist near-fields of orders $3^2, 5^2, 7^2, 17^2$ that are not fields. Every zero-symmetric near-ring with 1, whose elements satisfy $x^k = x$ for a fixed integer $k > 1$, is a subdirect product of near-fields satisfying the same equation (see [4] or the corresponding result for rings by Jacobson [2]). Hence, by Corollary 1.2, every zero-symmetric near-ring with 1 that satisfies $x^{2^m \cdot 9+1} = x$ for

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some natural number m is a subdirect product of finite near-fields. In particular addition is commutative for such a near-ring. We note that there exists a near-field N of size 9 whose multiplicative group is isomorphic to the quaternion group and consequently has exponent 4. Hence N is an example of a near-ring that satisfies $x^{4k+1} = x$ (for each natural number k) and whose multiplication is not commutative. However, by Corollary 1.2, every zero-symmetric near-ring with 1 that satisfies $x^{19} = x$ is a subdirect product of finite fields, and hence both addition and multiplication are commutative. This generalizes a result from [5].

2. Proofs

For the proof of Theorem 1.1 we will use the following results.

Fact 2.1. [8, Theorem 1] *Let V be an abelian group, and let G be a periodic group of automorphisms of V. If G is generated by elements of order 3 and G acts freely on V, then G is either cyclic or isomorphic to* $\mathrm{SL}(2,3)$ *or* $\mathrm{SL}(2,5)$.

Fact 2.2. [7, 12.3.5, 12.3.6] *Every group of exponent 3 is nilpotent.*

Fact 2.3. [3, Theorem 2.1.b] *Let G be a periodic infinite group. If G contains an involution whose centralizer in G is finite, then G contains an infinite abelian subgroup.*

Lemma 2.4. *Let G be a $\{2,3\}$-group all of whose 2-subgroups are finite and all of whose finite 3-subgroups have order at most 3. Then G is finite.*

Proof: Let S be a maximal 2-subgroup of minimal order of G. We use induction on the size of S. If $|S| = 1$, then G has exponent 3 by assumption. By Fact 2.2 and the assumption that all finite subgroups of G have size at most 3, we obtain that G has size at most 3.

Next we assume that $|S| > 1$. Let h be a central involution in S. We first show that $C_G(h)$ is finite. We claim that $\bar{C} := C_G(h)/\langle h\rangle$ satisfies the assumptions of the lemma. Certainly $\bar{C}$ is a $\{2,3\}$-group, all its 2-subgroups are finite, and all its finite 3-subgroups have size at most 3. Since $S/\langle h\rangle$ is a maximal 2-subgroup of $\bar{C}$ and $|S/\langle h\rangle| < |S|$, the group $\bar{C}$ is finite by the induction hypothesis. Hence $C_G(h)$ is finite. By the assumptions G is a periodic group all of whose abelian subgroups are finite. Thus G is not infinite by Fact 2.3. □

Fact 2.5. [3, Corollary 2.5] *Every infinite 2-group contains an infinite abelian subgroup.*

Lemma 2.6. *Let G be a finite group acting freely on an abelian group V. Then G is isomorphic to a Frobenius complement in some finite Frobenius group.*

Proof: First we assume that there exists $v \in V \setminus \{0\}$ of finite order. Then $W := \langle v^g \mid g \in G\rangle$ is a finitely generated abelian group of finite exponent. Hence W is finite and $W \rtimes G$ is a finite Frobenius group with complement G.

In the following we assume that V is torsionfree. Let $v \in V \setminus \{0\}$, and let $W := \langle v^g \mid g \in G\rangle$. Then W forms a torsionfree $\mathbb{Z}[G]$-module of finite rank, say r. In particular W determines a representation φ of G over $\mathbb{Q}$. We note that 1 is not an eigenvalue for any $\varphi(g)$ with $g \in G \setminus \{1\}$ since G acts freely on W. Now let p be a prime that does not divide the order of G. By [6, Theorem 15.11] there exists a finite field F of characteristic p and an F-representation φ^* corresponding to φ such that $\varphi^*(g)$ does not have eigenvalue 1 for any $g \in G \setminus \{1\}$. Thus $F^r \rtimes_{\varphi^*} G$ is a finite Frobenius group with complement G. □

Proof of Theorem 1.1: Let V and G satisfy the assumptions of the theorem. Let $T := \langle x \in G \mid x^3 = 1\rangle$. By Fact 2.1 the group T is finite with Sylow 3-subgroups of order at most 3. By its definition T is normal in G. We show that G/T satisfies the assumptions of Lemma 2.4. All finite 3-subgroups of G have exponent dividing 9 and are cyclic by Lemma 2.6 and [7, 10.5.6]. Hence the finite 3-subgroups of G/T have order at most 3.

Let S be a finite 2-subgroup of G. Then S is cyclic or a generalized quaternion group by Lemma 2.6 and [7, 10.5.6]. Since the exponent of S divides 2^m, we have $|S| \leq 2^{m+1}$. In particular all finite 2-subgroups of G/T have order at most 2^{m+1}. Hence G/T has no infinite abelian 2-subgroup. By Fact 2.5 all 2-subgroups of G are finite. Hence G/T satisfies the assumptions of Lemma 2.4 and therefore G/T is finite. So G is finite. □

For the proof of Corollary 1.2 we will use the following lemma.

Lemma 2.7. *Let p be a prime, let k, m, n be natural numbers with $n \leq 2$. If $p^k - 1 = 2^m \cdot 3^n$, then $p^k \in \{2^2, 3^2, 5^2, 7^2, 17^2\}$ or $k = 1$.*

Proof: If $m = 0$, then we obtain $p^k = 2^2$. For the following we assume $m > 0$. Then p is odd. First we suppose that k is even. Since $\gcd(p^{k/2} - 1, p^{k/2} + 1) = 2$, either $p^{k/2} - 1$ or $p^{k/2} + 1$ is a power of 2. In the former case we have $p^{k/2} + 1 = 2 \cdot 3^n$ with $0 \leq n \leq 2$. Thus $p \in \{5, 17\}$ and $k = 2$. In the latter case $p^{k/2} - 1 = 2 \cdot 3^n$ with $0 \leq n \leq 2$ yields $p \in \{3, 7\}$ and $k = 2$.

Next we assume that k is odd. Then $\frac{p^k - 1}{p - 1} = \sum_{i=0}^{k-1} p^i$ is odd and divides 9. This yields $k = 1$. □

Proof of Corollary 1.2: Let F be a near-field whose multiplicative group F^* has exponent $2^m \cdot 3^n$ for $0 \leq m, 0 \leq n \leq 2$. Since F^* acts freely on the additive group of F by multiplication, F^* is finite by Theorem 1.1. The Sylow 2-subgroup of F^* is cyclic or generalized quaternion, and the Sylow 3-subgroup of F^* is cyclic by [7, 10.5.6]. Hence we have $|F^*| \in \{2^m \cdot 3^n, 2^{m+1} \cdot 3^n\}$. Since the order of F is a prime power, F has prime order or $|F| \in \{2^2, 3^2, 5^2, 7^2, 17^2\}$ by Lemma 2.7. If $|F|$ is a prime or $|F| = 4$, then F^* is cyclic and hence F is a field. □

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