FIXED-POINT-FREE REPRESENTATIONS OVER FIELDS OF PRIME CHARACTERISTIC

PETER MAYR

ABSTRACT. The irreducible representations over prime fields of characteristic p for those solvable groups which occur as groups of fixed-point-free automorphisms on finite groups are determined.

1. INTRODUCTION

A group of automorphisms Φ is said to act fixed-point-free on a finite group G iff $|\Phi| > 1$ and no automorphism in Φ except the identity mapping fixes an element of G distinct from the group identity. Fixed-point-freeness poses strong conditions on both groups Φ and G. In particular, G has to be nilpotent. All subgroups of Φ of order pq with primes p and q are cyclic (see [Rob96], [Wol67], and others.)

The classification of the groups that occur as fixed-point-free automorphism groups is initially due to Zassenhaus, [Zas36] and [Zas85]. Nevertheless, while Zassenhaus determined presentations of these group, the question, in which way such a group acts on another was not treated. In particular, up to now only few results are known concerning, given a nilpotent group G, what are the actual subgroups of the automorphism group of G, which act fixed-point-free on G (see [KK95].)

The search for pairs of groups G and Φ such that Φ is fixed-point-free on G, has applications in the construction of certain geometries and designs, [Cla92], for the classification of Frobenius-groups and for the classification of finite simple nearrings. As a first attempt we will describe the groups acting fixed-point-free on elementary abelian groups, i.e., acting fixed-point-free on vectorspaces over $GF(p)$. Representation theory is the natural tool for this task.

Definition 1. If π is a representation of a group Φ, and if $\pi(\varphi)$ does not have 1 as an eigenvalue for all $\varphi \in \Phi \setminus \{\mathrm{id}_\Phi\}$, then π is said to be *fixed-point-free.*

Let π be a fixed-point-free representation of degree e over a field F. Let V be the e-dimensional vectorspace over F with some fixed basis B. Then $\pi(\Phi)$ acts fixed-point-free on V, by identifying $\pi(\varphi)$ for $\varphi \in \Phi$ with the V-isomorphism $x \mapsto \pi(\varphi) \cdot x$ with respect to this basis B.

Some simple facts on fixed-point-free representations are stated without proof: Each fixed-point-free representation is faithful. By Maschke, each fixed-point-free representation is a sum of irreducible fixed-point-free representations. Conversely, a sum of fixed-point-free representations is again fixed-point-free.

This work has been supported by the Austrian National Science Foundation (Fonds zur Förderung der wiss. Forschung) under Grant P12911-INF.

For this note let F denote the prime field of characteristic p, i.e. $GF(p)$, and let $\bar{F}$ denote the algebraic closure of F.

The determination of fixed-point-free representations over an algebraic closed field resembles closely the construction of fixed-point-free representations over the complex numbers as in [Wol67]. The fixed-point-free representations over F can then be obtained from those over $\bar{F}$ by methods as explained in [Isa94].

Let $\mathrm{Fpf}_E(\Phi)$ denote the set of irreducible fixed-point-free representations of Φ over the field E.

We follow the classification of the finite solvable fixed-point-free groups as given in [Wol67] and [Wäh87] and stick to the notation of groups of type I, II, III or IV as used there.

Functions for the computation of the $GF(p)$-representations given in this note are implemented as part of SONATA, [Tea00]. Moreover, functions for the computation of fixed-point-free automorphism groups on a given nilpotent group are available there.

Throughout this note, let the subgroup of the multiplicative group of prime residues modulo m generated by the integer x be denoted as $\langle x \rangle_m$.

If ρ is a representation of H a subrgoup of G, then the by ρ induced representation on G is denoted by $\mathrm{Ind}_H^G \rho$.

2. Cyclic Groups

The most simple fixed-point-free representations are those of cyclic groups.

Proposition 1. Let $\langle \alpha \rangle$ be a cyclic group of order m coprime to p. Then the irreducible, faithful representations of $\langle \alpha \rangle$ over $\bar{F}$ are of degree 1 and given by

$$\sigma_i(\alpha) = (a^i)$$

with a a primitive m-th root of unity and $\gcd(i, m) = 1$. There are $\phi(m)$ nonequivalent representations and each of them is fixed-point-free.

Proof. Straightforward. □

Proposition 2. Let $\langle \alpha \rangle$ be a cyclic group of order m coprime to p. Let σ be an irreducible, faithful representations of $\langle \alpha \rangle$ over F. Let e be the multiplicative order of p modulo m. Then

(a) $\sigma = \sigma_i \oplus \sigma_{ip} \oplus \cdots \oplus \sigma_{ip^{e-1}}$ over $\bar{F}$ for some i where σ_i is defined as in 1.
(b) There are $\phi(m)/e$ representations in $\mathrm{Fpf}_F(\langle \alpha \rangle)$ and all of them have degree e.

Proof. Suppose that σ_i for some i is an irreducible constituent of σ over $\bar{F}$. The Galois conjugacy class over F for the character of σ_i is given by the characters of $\{\sigma_i, \sigma_{ip}, \ldots, \sigma_{ip^{e-1}}\}$. Now by [Isa94] (9.21) σ splits over $\bar{F}$ as in (a) and the assertions on the degree of σ and the size of $\mathrm{Fpf}_F(\langle \alpha \rangle)$ in (b) follow. □

3. Metacyclic Groups

Let Φ be a metacyclic group, i.e., Φ has a cyclic normal subgroup with cyclic factor, admitting a fixed-point-free representation. Then by [Wol67] (5.5.1) Φ has generators α, β with relations $\alpha^m = 1, \beta^n = \alpha^{m'}, \beta^{-1}\alpha\beta = \alpha^r$ where n is the multiplicative order of r in $\mathbb{Z}_m^*$, the group of prime residues modulo m. Additionally, m' divides m, each prime divisor of n divides m/m' and $r = 1 \mod (m/m')$.

Let the group $\langle \alpha, \beta \rangle$ with the above presentation be denoted as $\Phi_{mc}(m, r)$. Then the size of $\Phi_{mc}(m, r)$ equals mn.

This presentation is not exactly the same but equivalent to the usual one given by Zassenhaus, Wolf, etc, for the groups having cyclic p-Sylow subgroups and admitting a fixed-point-free representation. The representations of $\Phi_{mc}(m, r)$ are more easily expressed using this presentation and parameters m and r than in terms of the original one.

We only state that the derived subgroup of $\Phi_{mc}(m, r)$ is $\Phi_{mc}(m, r)' = \langle \alpha^{m/m'} \rangle$ and the center is $C(\Phi_{mc}(m, r)) = \langle \beta^n \rangle = \langle \alpha^{m'} \rangle$. Thus $\langle \alpha \rangle = \langle \Phi_{mc}(m, r)', C(\Phi_{mc}(m, r)) \rangle$ is certainly fully invariant and the unique maximal cyclic subgroup of $\Phi_{mc}(m, r)$.

Not only groups with cyclic p-Sylow subgroups can be expressed in the form $\Phi_{mc}(m, r)$ but for feasible choice of the parameters m, r also certain groups with the 2-Sylow subgroup being a quaternion group can be described in that way. In the diction of [Wol67] and [Wäh87] these groups are of type II.

Proposition 3. Let $\Phi_{mc}(m, r) = \langle \alpha, \beta \rangle$ and $H = \langle \alpha \rangle$.

(a) The irreducible, fixed-point-free representations of $\Phi = \Phi_{mc}(m, r)$ over $\bar{F}$ have degree $[\Phi_{mc}(m, r) : \langle \alpha \rangle] = n$ and are given by the induced representations

$$\pi_i := Ind_H^{\Phi} \sigma_i$$

with σ_i an irreducible fixed-point-free representation of H as defined in Proposition 1.

(b) Two representations π_i and π_j are equivalent iff $ij^{-1} \in \langle r \rangle_m$, i.e., iff there exists an integer f such that $i = jr^f \mod m$.

(c) There are $\phi(m)/n$ representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_{mc}(m, r))$.

Proof. (a) The restriction of an irreducible fixed-point-free representation π of Φ to H is a sum of irreducible fixed-point-free representations of the cyclic group H as determined in Proposition 2.

Let σ_i be one constituent of $\pi|_H$ and let $\sigma_i' : x \mapsto \sigma_i(\beta^{-1} x \beta)$ denote the representation conjugate by β. Then σ_i' is equivalent to σ_{ri} but not to σ_i if $r \neq 1 \mod m$. Thus, by [AW92] (8.5.11) σ_i induces an irreducible representation. Moreover, $\pi = \mathrm{Ind}_H^{\Phi} \sigma_i$ is fixed-point-free: each prime dividing n also divides $\mathrm{ord}\, \beta/n$ and thus the elements of $\langle \alpha, \beta \rangle$ of prime order are elements of $\langle \alpha \beta$. Suppose that for some $\varphi = \alpha^k \beta^l \in \Phi \setminus H$ the matrix $\pi(\alpha^k \beta^l)$ has eigenvalue 1. Then all powers of $\alpha^k \beta^l$ are represented as matrices with 1 as eigenvalue in contradiction to the fact that the powers of prime order are in H and that $\pi|_H = \sigma_i \oplus \sigma_{ir} \oplus \cdots \oplus \sigma_{ir^{n-1}}$ is fixed-point-free.

(b) and (c) follow from the fact that two induced representations $\pi_i = \mathrm{Ind}_H^{\Phi} \sigma_i$ and $\pi_j = \mathrm{Ind}_H^{\Phi} \sigma_j$ are equivalent if and only if σ_i and σ_j are conjugate by some element in Φ. □

Inducing the irreducible fixed-point-free representations of $\langle \alpha \rangle \leq \Phi_{mc}(m, r)$ over F as given in Proposition 2 will in general not give an irreducible representation of $\Phi_{mc}(m, r)$ over F, because for $\sigma \in \mathrm{Fpf}_F(\langle \alpha \rangle)$ a conjugate by some power of β may actually be equivalent to σ again.

Nevertheless, it is possible to determine the constituents of $\sigma \in \mathrm{Fpf}_F(\Phi_{mc}(m, r))$ over $\bar{F}$.

Proposition 4. Let π be an irreducible, fixed-point-free representation of $\Phi_{mc}(m,r)$ over F. Let $t := [\langle r,p\rangle_m : \langle p\rangle_m]$.

(a) Then $\pi = \pi_i \oplus \pi_{ip} \oplus \cdots \oplus \pi_{ip^{t-1}}$ over $\bar{F}$ for some i where π_i is defined as in Proposition 3.

(b) Two irreducible fixed-point-free F-representations π and π' with $\bar{F}$-constituents π_i and π_j, respectively, are equivalent iff $ij^{-1} \in \langle r,p\rangle_m$.

(c) There are $\phi(m)/(nt)$ representations in $\mathrm{Fpf}_F(\Phi_{mc}(m,r))$ and all of them have degree nt.

Proof. Following the same ideas as the proof of Proposition 2. □

4. Groups of type II

A group Φ admitting a fixed-point-free representation is said to be of type II if the 2-Sylow subgroups of Φ are generalized quaternion groups and Φ has a subgroup H of index 2 such that all the p-Sylow subgroups of H are cyclic. Presentations of groups of type II are given in [Wol67], 6.1.11.

Φ has generators α, β, q with $H = \langle \alpha, \beta\rangle$ fulfilling the relations as given for the metacyclic group in the previous section. Additionally, $q^4 = 1, q^2 \in H \cap C(\Phi)$ and q normalizes $\langle\alpha\rangle$ and $\langle\beta\rangle$, that is, there are integers l,k such that $q^{-1}\alpha q = \alpha^k$ and $q^{-1}\beta q = \beta^l$. Since $\beta^{-1}q^{-1}\beta q = \beta^{l-1}$ is an element of the cyclic derived subgroup Φ' of Φ, β^{l-1} commutes with α. Thus, $\beta^{l-1} \in \langle\alpha\rangle$ and n divides $l-1$. In particular, for p an odd prime divisor of n and t maximal such that p^t divides $\mathrm{ord}\,\beta = nm/m'$, it holds that $l = 1 \mod p^t$. On the other hand, with s maximal such that 2^s divides $\mathrm{ord}\,\beta$, we have that $l = -1 \mod 2^s$. Otherwise β and q commute, Φ has cyclic 2-Sylow subgroups and is metacyclic of type I. Thus the parameter l is completely determined by the pair of congruences $l = -1 \mod 2^s$ and $l = 1 \mod (nm)/(2^s m')$.

We note that because $\beta^n = \alpha^{m'}$ also $k = l \mod (m/m')$, posing a condition on k.

Let $\beta^{-1}\alpha\beta = \alpha^r$ and $\mathrm{ord}\,\alpha = m$. Assume that $k \in \langle r\rangle_m$. Note that k is of multiplicative order 2 since $q^2 \in C(\Phi)$ and r has order n. This implies that n is even and $k = r^{n/2} \mod m$. Now $\beta^{n/2}q^{-1}$ centralizes α. Therefore $G = \langle\alpha, \beta^{n/2}q^{-1}\rangle$ is a cyclic subgroup of index n in Φ. Moreover, G is normal in Φ and the factor Φ/G is generated by βG, hence cyclic.

Thus, if $k \in \langle r\rangle_m$, then Φ is actually metacyclic and the representations of Φ are determined in the previous section.

We restrict ourselves to the remaining case that $k \notin \langle r\rangle_m$.

With feasible choice of the parameters m,r,k the presentation of Φ of type II is already determined. We introduce the notation $\Phi_2(m,r,k)$ for the group $\langle\alpha,\beta,q\rangle$, not metacyclic, with $\alpha^m = 1, \beta^n = \alpha^{m'}, \beta^{-1}\alpha\beta = \alpha^r$ where n is the multiplicative order of r in $\mathbb{Z}_m^*$ and $q^{-1}\alpha q = \alpha^k, q^{-1}\beta q = \beta^l$ with $k^2 = 1 \mod (mn)$ and $l = -1 \mod 2^s$ and $l = 1 \mod (nm)/(2^s m')$ where s is maximal such that 2^{s-1} divides m.

Additionally, m' divides m and $\gcd(m/m', m') = 1$, each prime divisor of n divides m/m' and $r = 1 \mod (m/m')$. We have that $k = l \mod (m/m')$, $k \notin \langle r\rangle_m$ and n is 2 times an odd integer. The size of $\Phi_2(m,r,k)$ equals $2mn$. $\langle\alpha\rangle$ is the unique maximal cyclic normal subgroup in $\Phi_2(m,r,k)$ and includes both centre and the derived subgroup.

Let μ be an irreducible representation of Φ with $\pi_i \in \mathrm{Fpf}_{\bar{F}}(\langle\alpha,\beta\rangle)$ as determined in Proposition 3 an irreducible constituent of the restriction of μ to $\langle\alpha,\beta\rangle$. Now the conjugate representation $\pi_i' : x \mapsto \pi(q^{-1}xq)$ is equivalent to π_{ik} but not to π_i. Thus the restriction of μ to $\langle\alpha,\beta\rangle$ is not isotypic and by [AW92], (5.13), μ is induced by π_i.

Hence we may state:

Proposition 5. Let $\Phi_2(m,r,k) = \langle\alpha,\beta,q\rangle$ be of type II.

(a) The irreducible, fixed-point-free representations of $\Phi = \Phi_2(m,r,k)$ over $\bar{F}$ have degree $[\Phi : \langle\alpha\rangle] = 2n$ and are given by the induced representations

$$\mu_i := Ind^{\Phi}_{\langle\alpha,\beta\rangle}\pi_i$$

with π_i an irreducible fixed-point-free representation of $\langle\alpha,\beta\rangle$ as defined in Proposition 3.

(b) Two representations μ_i and μ_j are equivalent iff $ij^{-1} \in \langle k,r\rangle_m$.

(c) There are $\phi(m)/2n$ representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_2(m,r,k))$.

Once again the F-representations are a bit more complicated to determine. If $k \notin \langle p,r\rangle_m$, then the same argument as above gives the elements of $\mathrm{Fpf}_F(\Phi_2(m,r,k))$ as inductions of the representations in $\mathrm{Fpf}_F(\langle\alpha,\beta\rangle)$.

For $k \in \langle p,r\rangle_m$, i.e., $k = p^{e/2}$ or $k = p^{e/2}r^{n/2}$ with e the multiplicative order of p in $\mathbb{Z}_m^*$, the restriction of an irreducible fixed-point-free F-representation to $\langle\alpha,\beta\rangle$ is again irreducible.

Proposition 6. Let μ be an irreducible, fixed-point-free representation of $\Phi_2(m,r,k) = \langle\alpha,\beta,q\rangle$ of type II, not metacyclic, over F. Let $s := [\langle r,k,p\rangle_m : \langle r,p\rangle_m]$.

(a) Then $\mu = \mu_i \oplus \mu_{ip} \oplus \cdots \oplus \mu_{ip^{s-1}}$ over $\bar{F}$ for some i where μ_i is defined as in Proposition 5.

(b) Two irreducible fixed-point-free F-representations μ and μ' with $\bar{F}$-constituents μ_i and μ_j, respectively, are equivalent iff $ij^{-1} \in \langle r,k,p\rangle_m$.

(c) There are $\phi(m)/(ns)$ representations in $\mathrm{Fpf}_F(\Phi_2(m,r,k))$ and all of them have degree ns.

Proof. Similar to the proof of Proposition 2. □

5. Groups of type III

A group Φ admitting a fixed-point-free representation is said to be of type III if Φ has a normal subgroup isomorphic to some binary tetrahedral group $T_\nu = \langle p,q,r\rangle$ ($\langle p,q\rangle$ is the quaternion of size 8 and $r^{-1}pr = q, r^{-1}qr = pq, r^{3^\nu} = 1$). Additionally, 16 must not divide $|\Phi|$.

Hence, Φ is isomomorphic to a semidirect product of a binary tetrahedral group of size $8 * 3^\nu$ with $\nu > 0$ and a metacyclic group H of type I of order coprime to 2 and 3.

Let the group $\langle p,q,\alpha,\beta\rangle$ of type III with the above presentation be denoted as $\Phi_3(m,r)$. The size of $\Phi_3(m,r)$ equals $8mn$.

For the characterization of the fixed-point-free representations of groups of type III, the fixed-point-free representations of the binary tetrahedral groups are of importance.

First of all we consider the fixed-point-free representations of Q_8, the quaternion group of order 8 over F. There is at least one irreducible representation of Q_8 over $GF(p)$ with odd p of degree greater than 1, since Q_8 is not abelian. By cardinality reasons there is only one such representation and its degree equals 2. Now, for integers u, v such that $u^2 + v^2 = -1 \mod p$

$$\rho(p) = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}, \rho(q) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

defines this up to equivalence unique representation of Q_8 and ρ is fixed-point-free for all p odd (see [May], Proposition 5.19). Let in the following the matrix $\rho(p)$ be denoted by P and $\rho(q) =: Q$.

Let $T := T_1$ denote the binary tetrahedral group of size 24 and let τ be a fixed-point-free $\bar{F}$-representation of T. Clearly, the reduction of τ to $\langle p, q\rangle \leq T$ is a multiple of ρ. In particular τ has even degree. The factor $T/C(T)$ is isomorphic to the alternating group A_4 of size 12. The squares of the degrees of the $\bar{F}$-representations of T which do not map $C(T)$ to the identity matrix I sum up to 12. Thus by cardinality reasons there are 3 such representations, each of degree 2.

With $\tau_i|_{\langle p,q\rangle} = \rho$ and

$$\tau_i(r) = \frac{1}{2}\begin{pmatrix} -1+u+v & -1-u+v \\ 1-u+v & -1-u-v \end{pmatrix} * b^i$$

with b a primitive third root of identity over $\bar{F}$ we have found these three nonequivalent faithful irreducible representations of T as τ_i where $0 \leq i \leq 2$. Let the matrix $\tau_0(r)$ be denoted by R. The determinant of R equals 1 and its order is 3. Thus R has eigenvalues b, b^2. Consequently, the representations τ_1 and τ_2 are not fixed-point-free, while τ_0 is fixed-point-free and even an F-representation.

We note that τ_0 is the unique irreducible fixed-point-free F-representation of the binary tetrahedral group T.

The nonequivalent irreducible fixed-point-free $\bar{F}$-representations of T_ν are given by

$$\tau_i(p) = P, \tau_i(q) = Q, \tau_i(r) = R * b^i$$

for $1 \leq i \leq 3^\nu, \gcd(i, 3^\nu) = 1$ and b a primitive (3^ν)-th root of identity over $\bar{F}$. These $2 * 3^{\nu-1}$ representations are also F-representations if and only if $b \in F$, that is, 3^ν divides $p-1$. In general, an irreducible fixed-point-free F-representation has the $\bar{F}$-constituents $\tau_i, \tau_{ip}, \dots, \tau_{ip^{e-1}}$ where $e = |\langle p\rangle_{3^\nu}|$, see Proposition 1. Thus an arbitrary irreducible fixed-point-free F-representation τ is of the form

$$\tau(p) = P \otimes \sigma(1), \tau(q) = Q \otimes \sigma(1), \tau(r) = R \otimes \sigma(r)$$

with $\sigma \in \mathrm{Fpf}_F(\langle r\rangle)$ an irreducible fixed-point-free F-representation of $\langle r\rangle$ of size 3^ν as determined in Proposition 1. Clearly, $\deg\tau = 2e$ and $|\mathrm{Fpf}_F(T_\nu)|$ for $\nu > 1$ equals $|\mathrm{Fpf}_F(C_{3^\nu})| = 2 * 3^{\nu-1}/e$.

Proposition 7. Let $\Phi = \Phi_3(m, r) = \langle p, q, \alpha, \beta\rangle$.

(a) If $3 \not| n$ and $\nu = 1$, then $\Phi = T \times \langle\alpha^3, \beta\rangle$ and the irreducible, fixed-point-free representations of Φ over $\bar{F}$ are of the form

$$\nu_i := \tau_0 \otimes \pi_i$$

with τ_0 the unique irreducible fixed-point-free representation of T and $\pi_i \in \mathrm{Fpf}_{\bar{F}}(\langle\alpha^3, \beta\rangle)$.

There are $\phi(m)/2n$ nonequivalent representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_3(m, r))$ and each has degree $2n$.

(b) If $3 \not| n$ and $\nu > 1$, then $\Phi = T_\nu \times \langle \alpha^{3^\nu}, \beta \rangle$ and the irreducible, fixed-point-free representations of Φ over $\bar{F}$ are of the form

$$\nu_{j,i} := \tau_j \otimes \pi_i$$

with $\tau_j \in \mathrm{Fpf}_{\bar{F}}(T_\nu)$ and $\pi_i \in \mathrm{Fpf}_{\bar{F}}(\langle \alpha^{3^\nu}, \beta \rangle)$.

There are $\phi(m)/n$ nonequivalent representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_3(m, r))$ and each has degree $2n$.

(c) If $3|n$, then $H = \langle p, q \rangle \times \langle \alpha \rangle$ is a normal subgroup of Φ of index n and the irreducible, fixed-point-free representations of Φ over $\bar{F}$ are of the form

$$\nu_i := \mathrm{Ind}_H^\Phi(\rho \otimes \sigma_i)$$

with ρ the irreducible fixed-point-free representation of Q_8 and $\sigma_i \in \mathrm{Fpf}_{\bar{F}}(\langle \alpha \rangle)$.

There are $\phi(m)/n$ nonequivalent representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_3(m, r))$ and each has degree $2n$.

Proof. If $3 \not| n$, then $\Phi_3(m, r)$ is the direct product of a binary tetrahedral group T_ν and a metacyclic group of order coprime to 6. Thus the irreducible $\bar{F}$-representations are tensor products of the irreducible $\bar{F}$-representations of the factors T_ν and $\langle \alpha^{3^\nu}, \beta \rangle$ and the tensor product of two representations is fixed-point-free if and only if both factor representations are fixed-point-free. This proves the form of the representations in (a) and (b). The assertions on the size of $\mathrm{Fpf}_{\bar{F}}(\Phi_3)$ follow from Proposition 3 and the numbers of fixed-point-free representations of the binary tetrahedral groups as determined above.

If 3 divides n, then α centralizes $\langle p, q \rangle$ and the irreducible fixed-point-free representations of $H = \langle p, q, \alpha \rangle$ are the tensor products of ρ and some σ_i as in Proposition 1. To show that the irreducible fixed-point-free representations of Φ are induced by $\rho \otimes \sigma_i$ for some i one follows the same argumentation as for the proof of Proposition 3. □

Similarily to the idea of characterizing the fixed-point-free F-representations of the binary tetrahedral group T_3^ν in terms of the fixed-point-free F-representations of the cyclic group of size 3^ν, we can determine the irreducible F-representations of an arbitrary group of type III in terms of the irreducible F-representation of the complements of Q_8.

Let I_2 denote the $2x2$ identity matrix over F.

Proposition 8. Let $\Phi_3(m, r) = \langle p, q, \alpha, \beta \rangle$.

(a) If $3 \not| n$ and $\nu = 1$, then the irreducible, fixed-point-free representations of $\Phi_3(m, r)$ over F are of the form

$$\nu := \tau_0 \otimes \pi$$

with τ_0 the unique irreducible fixed-point-free representation of T and $\pi \in \mathrm{Fpf}_F(\langle \alpha^3, \beta \rangle)$.

(b) If $3 \not| n$ and $\nu > 1$, then the irreducible, fixed-point-free representations of $\Phi_3(m, r)$ over F are given by

$$\nu(p) := P \otimes \pi(1), \nu(q) := Q \otimes \pi(1), \nu(\alpha) := R \otimes \pi(\alpha), \nu(\beta) := I_2 \otimes \pi(\beta)$$

with $\pi \in \mathrm{Fpf}_F(\langle \alpha, \beta \rangle)$.

(c) If $3|n$, then the irreducible, fixed-point-free representations of $\Phi_3(m, r)$ over F are given by

$$\nu(p) := P \otimes \pi(1), \nu(q) := Q \otimes \pi(1), \nu(\alpha) := I_2 \otimes \pi(\alpha), \nu(\beta) := R \otimes \pi(\beta)$$

with $\pi \in \mathrm{Fpf}_F(\langle \alpha, \beta \rangle)$.

6. Groups of type IV

A group $\Phi = \langle p, q, \alpha, \beta, z\rangle$ admitting a fixed-point-free representation is said to be of type IV, iff it has a normal subgroup $H = \langle p, q, \alpha, \beta\rangle$ of type III and index 2. Furthermore, the relations $z^2 = p^2, z^{-1}pz = qp, z^{-1}qz = q^{-1}$ are fulfilled and z normalizes both $\langle\alpha\rangle$ and $\langle\beta\rangle$. Let k, l be integers such that $z^{-1}\alpha z = \alpha^k$ and $z^{-1}\beta z = \beta^l$, respectively. Using $\beta^n = \alpha^{m'}$ we have that $\alpha^{rk} = z^{-1}\alpha^r z = z^{-1}\beta^{-1}\alpha\beta z = \beta^{-l}\alpha^k\beta^l = \alpha^{r^l k}$. Thus m divides $r^{l-1} - 1$ and n divides $l - 1$ implying furthermore that $l = 1 \mod \operatorname{ord}\beta$.

Hence, $z^{-1}\alpha z = \alpha^k$ and $z^{-1}\beta z = \beta$ with the condition that $k = 1 \mod (m/m'), k = -1 \mod 3^\nu$ where ν is maximal such that 3^ν divides m and $k^2 = 1 \mod m$. In particular, 3 does not divide n, a fact that is omitted in [Wol67].

We introduce the notation $\Phi_4(m, r, k)$ for the group $\langle p, q, \alpha, \beta, z\rangle$ with relations as above, determined by the parameters (m, r, k).

The minimal examples of groups of type IV are the binary octahedral groups $O_\nu = \Phi_4(3^\nu, 1, -1)$ generated by p, q, r, z where $\langle p, q, r\rangle = T_\nu$ of size $8 * 3^\nu$ and $z^2 = p^2, z^{-1}pz = qp, z^{-1}qz = q^{-1}, z^{-1}rz = r^{-1}$.

Let $O := O_1$ denote the binary octahedral group of size 48. O has 8 conjugacy classes. The factor $O/C(O)$ is isomorphic to the symmetric group S_4 of size 24, which has 5 conjugacy classes. The squares of the degrees of the 3 nonequivalent $\bar{F}$-representations of O which do not annihilate $C(O)$ sum up to 24. Thus there are 2 such representations of degree 2 and one of degree 4.

The faithful irreducible representation of degree 4 is given by the induction of one of the non fixed-point-free faithful representations of T as determined in the previous section and is certainly not fixed-point-free.

The reduction of a fixed-point-free $\bar{F}$-representation to $\langle p, q, r\rangle$ is a multiple of τ_0, the unique irreducible fixed-point-free F-representation of the binary tetrahedral group T. With $o_i|_{\langle p,q,r\rangle} = \tau_0$ and for $i = \pm 1$

$$o_i(z) = i\frac{1}{\sqrt{2}}\begin{pmatrix} u - v & u + v \\ u + v & -u + v \end{pmatrix}$$

we have found the 2 nonequivalent faithful irreducible representations of O and both are fixed-point-free. For further use let the matrix $\sqrt{2} * o_{+1}(z)$ be denoted by Z. o_{+1}, o_{-1} are F-representations iff 2 is a square in F. This holds iff 16 divides $p^2 - 1$ where p is the prime characteristic of F. If 16 does not divide $p^2 - 1$, then O of size 48 cannot act fixed-point-free on a vectorspace of dimension 2 over F anyway. In this case there is one fixed-point-free F-representation of degree 4 with $\bar{F}$-constituents o_{+1}, o_{-1}, equivalent to $\operatorname{Ind}_T^O(\tau_0)$.

The irreducible fixed-point-free $\bar{F}$-representations of O_ν with $\nu > 1$ are of the form

$$o_j = \operatorname{Ind}_{T_\nu}^{O_\nu} \tau_j$$

with $\tau_j \in \operatorname{Fpf}_{\bar{F}}(T_\nu)$. The degree of o_j equals 4. Since $\operatorname{Ind}_{T_\nu}^{O_\nu} \tau_j = \operatorname{Ind}_{T_\nu}^{O_\nu} \tau_{-j}$, there are in total $3^{\nu-1} = |\operatorname{Fpf}_{\bar{F}}(T_\nu)|/2$ non equivalent representations in $\operatorname{Fpf}_{\bar{F}}(Q_\nu)$.

Proposition 9. Let $\Phi = \Phi_4(m, r, k) = \langle p, q, \alpha, \beta\rangle$.

(a) If $k = 1 \mod (m/3^\nu)$, then $\Phi = O_\nu \times \langle \alpha^{3^\nu}, \beta\rangle$ and the irreducible, fixed-point-free representations of $\Phi_4(m,r,k)$ over $\bar{F}$ are of the form

$$\psi_{j,i} := o_j \otimes \pi_i$$

with $o_j \in \mathrm{Fpf}_{\bar{F}}(O_\nu)$ and $\pi_i \in \mathrm{Fpf}_{\bar{F}}(\langle \alpha^{3^\nu}, \beta\rangle)$.

If $9 \not| m$, then there are $\phi(m)/n$ nonequivalent representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_4(m,r,k))$ and each has degree $2n$.

If $9|m$, then there are $\phi(m)/2n$ nonequivalent representations in $\mathrm{Fpf}_{\bar{F}}(\Phi_4(m,r,k))$ and each has degree $4n$.

(b) If $k \neq 1 \mod (m/3^\nu)$, then $H = T_\nu \times \langle \alpha^{3^\nu}, \beta\rangle$ is of type III and the irreducible, fixed-point-free representations of Φ over $\bar{F}$ are induced by the irreducible, fixed-point-free representations $\nu_{j,i}$ of H.

If $9 \not| m$, then there are $\phi(m)/4n$ nonequivalent representations

$$\psi_i := \mathrm{Ind}_H^\Phi(\nu_i)$$

in $\mathrm{Fpf}_{\bar{F}}(\Phi_4(m,r,k))$ with $\nu_i \in \mathrm{Fpf}_{\bar{F}}(T \times \langle \alpha^3, \beta\rangle)$ and each has degree $4n$.

If $9|m$, then there are $\phi(m)/2n$ nonequivalent representations in

$$\psi_{j,i} := \mathrm{Ind}_H^\Phi(\nu_{j,i})$$

in $\mathrm{Fpf}_{\bar{F}}(\Phi_4(m,r,k))$ with $\nu_{j,i} \in \mathrm{Fpf}_{\bar{F}}(T_\nu \times \langle \alpha^{3^\nu}, \beta\rangle)$ and each has degree $4n$.

Proof. (a) is straightforward.

(b) As shown above $\Phi_4(m,r,k)$ has a subgroup of type III of index 2 and this subgroup is isomorphic to the direct product $H = T_\nu \times \langle \alpha^{3^\nu}, \beta\rangle$. The restriction of an irreducible fixed-point-free representation of Φ_4 to H has to be a sum of representations in $\mathrm{Fpf}_{\bar{F}}(H)$, which was given in Proposition 7 (a) and (b) respectively.

The conjugate of $\nu_{i,j}$ by z is equivalent to $\nu_{-i,kj}$ but not equivalent to $\nu_{i,j}$, since $k \neq 1 \mod (m/3^\nu)$. Thus by [AW92], (8.5.13), $\nu_{i,j}$ induces an irreducible representation. The number of nonequivalent inductions equals half the number of representations in $\mathrm{Fpf}_{\bar{F}}(H)$. □

Proposition 10. Let $\Phi_4(m,r,k) = \langle p, q, \alpha, \beta, z\rangle$ and let $H = \langle p, q, \alpha^{m/3^\nu}\rangle \times \langle \alpha^{3^\nu}, \beta\rangle$. Then the irreducible, fixed-point-free representations of $\Phi_4(m,r,k)$ over F are those induced by the irreducible, fixed-point-free representations of H over F except in the following cases:

(a) If $m \mod 9 \neq 0$ and $k = 1 \mod (m/3)$ and $16|p^2-1$, then the irreducible, fixed-point-free representations of $\Phi_4(m,r,k)$ over F are given by the tensor products

$$o_j \otimes \pi_i$$

for $o_{+1}, o_{-1} \in \mathrm{Fpf}_F(O)$ and $\pi_i \in \mathrm{Fpf}_F(\langle \alpha^3, \beta\rangle)$.

(b) If $m \mod 9 \neq 0$ and $k = 1 \mod (m/3)$ and $16 \not| p^2-1$ but $2|e$, the degree of the fixed-point-free representations of $Z_{m/3}$, then the irreducible, fixed-point-free representations of $\Phi_4(m,r,k)$ over F are given by

$$\psi(p) := P \otimes \pi_i(1), \psi(q) := Q \otimes \pi_i(1),$$

$$\psi(\alpha) := R \otimes \pi_i(\alpha^3), \psi(\beta) := I_2 \otimes \pi_i(\beta), \psi(z) := Z \otimes C$$

where $\pi_i \in \mathrm{Fpf}_F(\langle \alpha^3, \beta\rangle)$ and C commutes with $\pi_i(\alpha^3)$ and $\pi_i(\beta)$, $C^2 = \frac{1}{2}\pi_i(1)$.

(c) If $m \mod 9 \neq 0$ and $2|e$ and $k = p^{e/2} \mod (m/3)$, then the irreducible, fixed-point-free representations of Φ_4 over F are given by

$$\psi(p) := P \otimes \pi_i(1), \psi(q) := Q \otimes \pi_i(1),$$

$$\psi(\alpha) := R \otimes \pi_i(\alpha^3), \psi(\beta) := I_2 \otimes \pi_i(\beta), \psi(z) := Z \otimes C$$

where $\pi_i \in \mathrm{Fpf}_F(\langle \alpha^3, \beta \rangle)$. $C^{-1}\pi_i(\alpha^3)C = \pi_i(\alpha^3)^{p^{e/2}}$ and C commutes with $\pi_i(\beta)$, $C^2 = \frac{1}{2}\pi_i(1)$.

(d) If $m \mod 9 = 0$ and $2|e$ and $k = p^{e/2} \mod m$, then the irreducible, fixed-point-free representations of Φ_4 over F are given by

$$\psi(p) := P \otimes \pi_i(1), \psi(q) := Q \otimes \pi_i(1),$$

$$\psi(\alpha) := R \otimes \pi_i(\alpha), \psi(\beta) := I_2 \otimes \pi_i(\beta), \psi(z) := Z \otimes C$$

where $\pi_i \in \mathrm{Fpf}_F(\langle \alpha, \beta \rangle)$. $C^{-1}\pi_i(\alpha)C = \pi_i(\alpha)^{p^{e/2}}$ and C commutes with $\pi_i(\beta)$, $C^2 = \frac{1}{2}\pi_i(1)$.

Proof. By Proposition 8 the irreducible fixed-point-free representations of $H \leq \Phi$ are of the form

$$\nu_i(p) = P \otimes \pi_i(1), \nu_i(q) = Q \otimes \pi_i(1), \nu_i(\beta) = I_2 \otimes \pi_i(\beta)$$

and $\nu(\alpha) = R \otimes \pi_i(\alpha^3)$ with $\pi_i \in \mathrm{Fpf}_F(\langle \alpha^3, \beta \rangle)$ for $\nu = 1$ and $\nu(\alpha) = R \otimes \pi_i(\alpha)$ with $\pi_i \in \mathrm{Fpf}_F(\langle \alpha, \beta \rangle)$ for $\nu > 1$.

Let $\nu_i' : x \mapsto \nu(z^{-1}xz)$ denote the conjugate representation. Then ν_i is equivalent to ν_i' iff π_i is equivalent to π_{ki}, that is, iff $k \in \langle r, p \rangle_{m/3}$ or $k \in \langle r, p \rangle_m$, respectively. This holds for $k = 1 \mod m/3$ or $k = p^{e/2} \mod m/3$ if $\nu = 1$ and for $k = p^{e/2} \mod m$ if $\nu > 1$. For all other cases the induction of an irreducible fixed-point-free F-representation of H is again irreducible and all irreducible fixed-point-free F-representation of Φ are obtained in this way by [AW92] (8.5.13). Note that ν_i and $\nu_i k$ give the same induced representation.

The exceptional cases have to be checked each one on its own to obtain the assertions given in the proposition. □

References

[AW92] William A. Adkins and Steven H. Weintraub. *Algebra. An Approach via Module Theory*, volume 136 of *Graduate Texts in Mathematics*. Springer Verlag, New York, 1992.

[Cla92] James R. Clay. *Nearrings. Geneses and Applications.* Oxford University Press, Oxford, New York, Tokyo, 1992.

[Isa94] I. Martin Isaacs. *Character theory of finite groups.* Dover Publications, New York, 1994.

[KK95] Wen-Fong Ke and Hubert Kiechle. *Characterization of Some Finite Ferrero Pairs*, volume 336 of *Mathematics and Its Applications*, pages 153–160. Kluwer Academic Publishers, Dordrecht, Boston, London, 1995. Proceedings of the Conference on Near-Rings and Near-Fields, Fredericton, New Brunswick, Canada, July 18-24, 1993.

[May] Peter Mayr. Finite fixed point free automorphism groups – diploma thesis. Available from: http://www.algebra.uni-linz.ac.at/ stein/.

[Rob96] Derek J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate Texts in Mathematics*. Springer Verlag, New York, second edition, 1996.

[Tea00] The SONATA Team. *SONATA: Systems Of Nearrings And Their Applications.* Universität Linz, Austria, 2000. Available from: http://www.algebra.uni-linz.ac.at/sonata/.

[Wäh87] Heinz Wähling. *Theorie der Fastkörper.* Thales Verlag, Essen, 1987.

[Wol67] Joseph Albert Wolf. *Spaces of Constant Curvature*. McGraw-Hill, New York, 1967.

[Zas36] Hans Zassenhaus. *Über endliche Fastkörper*, volume 11 of *Abhandlungen aus dem Mathematischen Seminar der Univesität Hamburg*, pages 187–220. 1936.

[Zas85] Hans Zassenhaus. *On Frobenius Groups I*, volume 8 of *Results in Mathematics*, pages 132–145. 1985.

Peter Mayr
Institut für Mathematik
Johannes Kepler Universität
A-4040 Linz
Austria
E-mail address: peter.mayr@algebra.uni-linz.ac.at